these methods for three-phase representation. The application of network matrices to short circuit calculations is presented in Chapter 6. Several methods are included and a typical computer program is described to illustrate a practical application of the techniques.

Chapter 7 contains a brief introduction to the solution of linear and nonlinear simultaneous algebraic equations. This material is presented in a manner that affords direct application to the solution of the load flow problem. The formulation and solution of the load flow problem is presented in Chapter 8. This chapter also describes the procedures for handling voltage-controlled buses, transformers, and tie line control. The different methods are compared from several points of view and a description is given of an actual program used for load flow calculations. In a manner similar to that in Chapter 7, Chapter 9 introduces methods for the numerical solution of the differential equations that are required for transient stability studies. Chapter 10 covers the formulation and solution techniques employed in transient studies and presents procedures for the detailed representation of synchronous and induction machines, exciter and governor systems, and the distance relays. An actual transient stability computer program is described.

The first efforts in the development of this material were made in the early 1950s at the American Electric Power Service Corporation as a result of the interest in the application of computers to the planning and operation of electric power systems. In 1959, the authors had an opportunity to work together as members of the staff of the American Electric Power Service Corporation and continued to work together on a part-time basis for several years. This made possible the further development of basic computer methods established in previous years.

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Minors and cofactors

The determinant obtained by striking out the ith row and jth column is called the *minor* of the element a_{ij}. Thus, for

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

the minor of a_{21} is

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

The order of this minor is one less than that of the original determinant. By striking out any two rows and columns a minor of order two less than the original determinant is obtained, etc.

The *cofactor* of an element is

$$(-1)^{i+j}(\text{minor of } a_{ij})$$

where the order of the minor of a_{ij} is $n - 1$. The cofactor of a_{21}, designated by A_{21}, is

$$A_{21} = (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

The following relationships between a determinant and cofactors exist:

1. The sum of the products of the elements in any row (or column) and their cofactors is equal to the determinant:

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \tag{2.3.2}$$

2. The sum of the products of the elements in any row (or column) and the cofactors of the corresponding elements in another row (or column) is equal to zero:

$$a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} = 0 \tag{2.3.3}$$

Adjoint

If each element of a square matrix is replaced by its cofactor and then the matrix is transposed, the resulting matrix is an *adjoint* which is designated by A^+

$$A^+ = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where A^+ is the adjoint of A. It should be noted that the elements of the adjoint A^+ are the cofactors of the elements of A, but are placed in transposed position. The matrix B is the inverse of A and is written A^{-1}.

Multiplying A by its inverse,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \frac{A_{31}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \frac{A_{32}}{|A|} \\ \frac{A_{13}}{|A|} & \frac{A_{23}}{|A|} & \frac{A_{33}}{|A|} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

results in the unit matrix. This follows from the relationships (2.3.2) and (2.3.3). A diagonal term of U, such as u_{11}, equals 1 since

$$a_{11}\frac{A_{11}}{|A|} + a_{12}\frac{A_{12}}{|A|} + a_{13}\frac{A_{13}}{|A|} = \frac{|A|}{|A|} = 1$$

and an off-diagonal term, such as u_{12}, equals zero since

$$a_{11}\frac{A_{21}}{|A|} + a_{12}\frac{A_{22}}{|A|} + a_{13}\frac{A_{23}}{|A|} = \frac{0}{|A|} = 0$$

Thus

$$AA^{-1} = A^{-1}A = U$$

To solve for X from the matrix equation (2.4.2) both sides of the equation are premultiplied by A^{-1}.

$$\begin{aligned} AX &= Y \\ A^{-1}AX &= A^{-1}Y \\ UX &= A^{-1}Y \\ X &= A^{-1}Y \end{aligned}$$

The order of the matrices in the product must be maintained.

If the determinant of a matrix is zero, the inverse does not exist. Such a matrix is called a *singular matrix*. If the determinant of a matrix is not zero, the matrix is a *nonsingular matrix* and has a unique inverse.

The inverse of the product of matrices can be obtained by the reversal rule, i.e.,

$$(AB)^{-1} = B^{-1}A^{-1}$$

The transpose and inverse operations on a matrix can be interchanged, i.e.,

$$(A^t)^{-1} = (A^{-1})^t$$

2.9 Given the partitioned matrix:

$$A = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & -2 & 1 \\ \hline -6 & 4 & 2 \end{array}\right]$$

Determine $B = A^{-1}$ using the formulas (2.4.3) for the inverse of a partitioned matrix.

2.10 Given:

$$\left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 3 & 4 \\ 2 & 6 & 0 & 0 & 1 \\ \hline 0 & 0 & 14 & 5 & 1 \\ 3 & 0 & 5 & 10 & 2 \\ 4 & 1 & 1 & 2 & 12 \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \\ 1 \\ 1 \end{array}\right]$$

Determine A_2.

2.11 Given the partitioned matrices:

$$A = \left[\begin{array}{cc|ccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 5 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

$$B = \left[\begin{array}{cc|ccc|c} 1 & 3 & 2 & 5 & 6 & 1 \\ 4 & 7 & 1 & 3 & 2 & 4 \\ \hline 3 & 4 & 2 & 6 & 5 & 1 \\ 4 & 6 & 3 & 1 & 2 & 5 \\ 2 & 6 & 7 & 3 & 8 & 1 \\ \hline 7 & 2 & 3 & 1 & 4 & 5 \end{array}\right]$$

Determine $C = AB$.

chapter 3
Incidence and network matrices

3.1 Introduction

The formulation of a suitable mathematical model is the first step in the analysis of an electrical network. The model must describe the characteristics of individual network components as well as the relations that govern the interconnection of these elements. A network matrix equation provides a convenient mathematical model for a digital computer solution.

The elements of a network matrix depend on the selection of the independent variables, which can be either currents or voltages. Correspondingly, the elements of the network matrix will be impedances or admittances.

The electrical characteristics of the individual network components can be presented conveniently in the form of a primitive network matrix. This matrix, while adequately describing the characteristics of each component, does not provide any information pertaining to the network connections. It is necessary, therefore, to transform the primitive network matrix into a network matrix that describes the performance of the interconnected network.

The form of the network matrix used in the performance equation depends on the frame of reference, namely, bus or loop. In the bus frame of reference the variables are the nodal voltages and nodal currents. In the loop frame of reference the variables are loop voltages and loop currents.

The formation of the appropriate network matrix is an integral part of a digital computer program for the solution of power system problems.

3.2 Graphs

In order to describe the geometrical structure of a network it is sufficient to replace the network components by single line segments irrespective of the characteristics of the components. These line segments are called *elements* and their terminals are called *nodes*. A node and an element are *incident* if the node is a terminal of the element. Nodes can be incident to one or more elements.

A *graph* shows the geometrical interconnection of the elements of a network. A *subgraph* is any subset of elements of the graph. A *path* is a subgraph of connected elements with no more than two elements connected to any one node. A graph is *connected* if and only if there is a path between every pair of nodes. If each element of the connected graph is assigned a direction it is then *oriented*. A representation of a power system and the corresponding oriented graph are shown in Fig. 3.1.

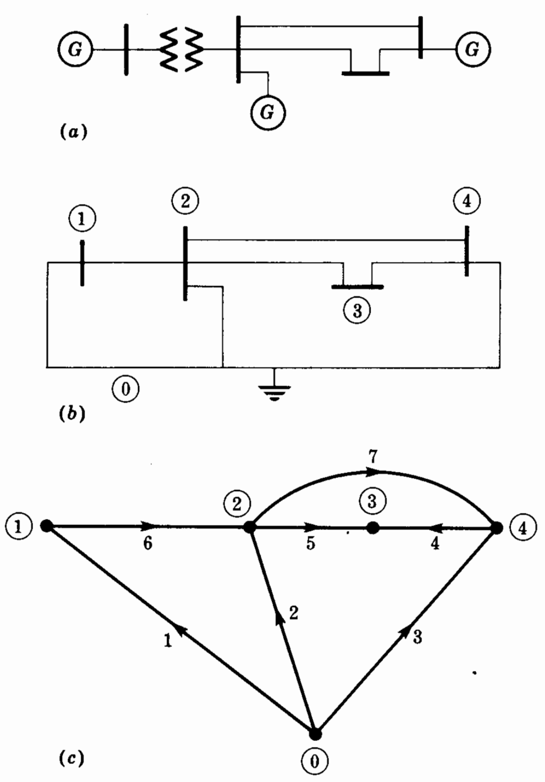


***Fig. 3.1** Power system representations. (a) Single line diagram; (b) positive sequence network diagram; (c) oriented connected graph.*

The basic loop incidence matrix, of dimension $e \times l$, for the graph shown in Fig. 3.3 is

$C =$

e \ l	Basic loops E	F	G
1		1	
2	1	−1	1
3	−1		−1
4	−1		
5	1		
6		1	
7			1

The matrix C can be partitioned into submatrices C_b and U_l where the rows of C_b correspond to branches and the rows of U_l to links. The partitioned matrix is

$C =$

e \ l	Basic loops E	F	G
1		1	
2	1	−1	1
3	−1		−1
4	−1		
5	1		
6		1	
7			1

$=$

e \ l	Basic loops
Branches	C_b
Links	U_l

The identity matrix U_l shows the one-to-one correspondence of links to basic loops.

Augmented loop incidence matrix $\hat{C}$

The number of basic loops in a connected graph is equal to the number of links. In order to have a total number of loops equal to the number of

It follows from equations (3.5.15) and (3.5.16) that

$$Y_{BR} = B^t[y]B$$

The basic cut-set matrix B is a singular matrix and therefore $B^t[y]B$ is a singular transformation of $[y]$.

The branch impedance matrix can be obtained from

$$Z_{BR} = Y_{BR}^{-1} = (B^t[y]B)^{-1}$$

Loop impedance and loop admittance matrices

The loop impedance matrix Z_{LOOP} can be obtained by using the basic loop incidence matrix C to relate the variables and parameters of the primitive network to loop quantities of the interconnected network. The performance equation of the primitive network

$$\bar{v} + \bar{e} = [z]\bar{\imath}$$

is premultiplied by C^t to obtain

$$C^t\bar{v} + C^t\bar{e} = C^t[z]\bar{\imath} \tag{3.5.17}$$

Since the matrix C shows the incidence of elements to basic loops, $C^t\bar{v}$ gives the algebraic sum of the voltages around each basic loop. In accordance with Kirchhoff's voltage law, the algebraic sum of the voltages around a loop is zero. Hence

$$C^t\bar{v} = 0 \tag{3.5.18}$$

Similarly $C^t\bar{e}$ gives the algebraic sum of the source voltages around each basic loop. Therefore

$$\bar{E}_{LOOP} = C^t\bar{e} \tag{3.5.19}$$

Since power is invariant

$$(\bar{I}^*_{LOOP})^t\bar{E}_{LOOP} = (\bar{\imath}^*)^t\bar{e}$$

Substituting for $\bar{E}_{LOOP}$ from equation (3.5.19), then

$$(\bar{I}^*_{LOOP})^tC^t\bar{e} = (\bar{\imath}^*)^t\bar{e}$$

Since this equation is valid for all values of $\bar{e}$, it follows that

$$(\bar{\imath}^*)^t = (\bar{I}^*_{LOOP})^tC^t$$

Hence,

$$\bar{\imath} = C^*\bar{I}_{LOOP}$$

Since C is a real matrix,

$$C^* = C$$

and

$$\bar{\imath} = C\bar{I}_{LOOP} \qquad (3.5.20)$$

Substituting from equations (3.5.18), (3.5.19), and (3.5.20) into (3.5.17) yields

$$\bar{E}_{LOOP} = C^t[z]C\bar{I}_{LOOP} \qquad (3.5.21)$$

The performance equation of the network in the loop frame of reference is

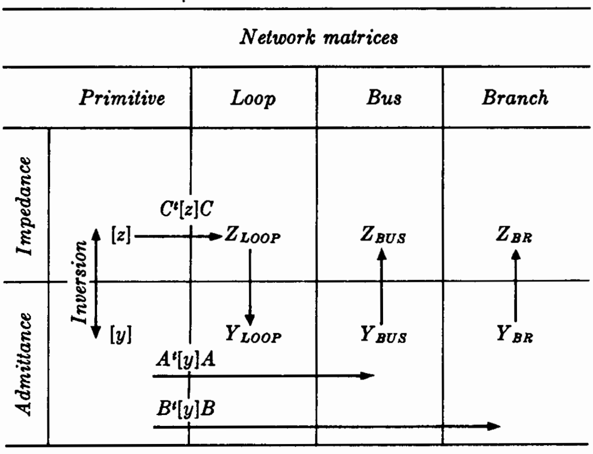
$$\bar{E}_{LOOP} = Z_{LOOP}\bar{I}_{LOOP} \qquad (3.5.22)$$

and it follows from equations (3.5.21) and (3.5.22) that

$$Z_{LOOP} = C^t[z]C$$

Since C is a singular matrix, $C^t[z]C$ is a singular transformation of $[z]$.

Table 3.1 Formation of network matrices by singular transformations



However,

$$C\bar{I}_{LOOP} = \hat{C}\hat{I}_{LOOP}$$

then

$$\bar{\imath} = \hat{C}\hat{I}_{LOOP} \tag{3.6.15}$$

Substituting from equation (3.6.15) into equation (3.6.14),

$$\hat{E}_{LOOP} = \hat{C}^t[z]\hat{C}\hat{I}_{LOOP} \tag{3.6.16}$$

Since the performance equation of the augmented network is

$$\hat{E}_{LOOP} = \hat{Z}_{LOOP}\hat{I}_{LOOP} \tag{3.6.17}$$

it follows from equations (3.6.16) and (3.6.17) that the impedance matrix of the augmented network is

$$\hat{Z}_{LOOP} = \hat{C}^t[z]\hat{C} \tag{3.6.18}$$

Equation (3.6.18) can be written in the partitioned form

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} U_b & 0 \\ C_b^t & U_l \end{bmatrix} \begin{bmatrix} z_{bb} & z_{bl} \\ z_{lb} & z_{ll} \end{bmatrix} \begin{bmatrix} U_b & C_b \\ 0 & U_l \end{bmatrix} \tag{3.6.19}$$

where $[z_{bb}]$ = primitive impedance matrix of branches
$[z_{bl}] = [z_{lb}]^t$ = primitive impedance matrix whose elements are the mutual impedances between branches and links
$[z_{ll}]$ = primitive impedance matrix of links

It follows from equation (3.6.19) that

$$Z_4 = C_b^t[z_{bb}]C_b + [z_{lb}]C_b + C_b^t[z_{bl}] + [z_{ll}] \tag{3.6.20}$$

Since

$$Z_{LOOP} = C^t[z]C$$

or

$$Z_{LOOP} = \begin{bmatrix} C_b^t & U_l \end{bmatrix} \begin{bmatrix} z_{bb} & z_{bl} \\ z_{lb} & z_{ll} \end{bmatrix} \begin{bmatrix} C_b \\ U_l \end{bmatrix}$$

then

$$Z_{LOOP} = C_b^t[z_{bb}]C_b + [z_{lb}]C_b + C_b^t[z_{bl}] + [z_{ll}] \tag{3.6.21}$$

From equations (3.6.20) and (3.6.21), therefore,

$$Z_{LOOP} = Z_4$$

Hence equation (3.6.33) becomes

$$Y_{BR} = KY_{BUS}K^t \qquad (3.6.34)$$

The branch impedance matrix is

$$Z_{BR} = Y_{BR}^{-1} = (K^t)^{-1}Y_{BUS}^{-1}K^{-1} \qquad (3.6.35)$$

From equation (3.3.2),

$$K^t = A_b^{-1} \qquad (3.6.36)$$

Substituting from equation (3.6.36) into equation (3.6.35),

$$Z_{BR} = A_b Z_{BUS} A_b^t$$

Derivation of bus admittance and impedance matrices from branch admittance and impedance matrices

Equation (3.6.34) is premultiplied by K^{-1} and postmultiplied by $(K^t)^{-1}$ to obtain

$$K^{-1}Y_{BR}(K^t)^{-1} = Y_{BUS} \qquad (3.6.37)$$

Substituting from equation (3.6.36) into equation (3.6.37),

$$Y_{BUS} = A_b^t Y_{BR} A_b$$

Since

$$Z_{BUS} = Y_{BUS}^{-1}$$

then

$$Z_{BUS} = (A_b^t Y_{BR} A_b)^{-1} \qquad \text{or} \qquad Z_{BUS} = K^t Z_{BR} K$$

The nonsingular transformations for obtaining network matrices are summarized in Table 3.3.

3.7 Example of formation of incidence and network matrices

The method of forming the incidence and network matrices will be illustrated for the network shown in Fig. 3.10. The incidence matrices for a given network are not unique and depend on the orientation of the graph and the selection of branches, basic cut-sets, and basic loops. However, the network matrices are unique.

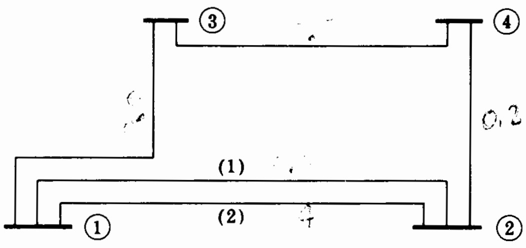


Fig. 3.10 ***Sample network.***

Problem

a. Form the incidence matrices $\hat{A}$, A, K, B, $\hat{B}$, C and $\hat{C}$ for the network shown in Fig. 3.10.
b. Form the network matrices Y_{BUS}, Y_{BR}, and Z_{LOOP} by singular transformations.
c. Form the network matrices Z_{LOOP}, Z_{BR}, and Z_{BUS} by nonsingular transformations.

Solution

The impedance data for the sample network is given in Table 3.4.

Table 3.4 ***Impedances for sample network***

	Self		*Mutual*	
Element number	*Bus code* p–q	*Impedance* $z_{pq,pq}$	*Bus code* r–s	*Impedance* $z_{pq,rs}$
1	1–2(1)	0.6		
2	1–3	0.5	1–2(1)	0.1
3	3–4	0.5		
4	1–2(2)	0.4	1–2(1)	0.2
5	2–4	0.2		

The network contains four nodes and five elements, that is, $n = 4$ and $e = 5$. The number of branches is

$$b = n - 1 = 3$$

and the number of basic loops is

$$l = e - n + 1 = 2$$

The basic and open loops of the oriented connected graph are shown in Fig. 3.13. The basic loop incidence matrix is

$C =$

e \ l	D	E
1	−1	1
2		−1
3		−1
4	1	
5		1

The augmented loop incidence matrix is

$\hat{C} =$

e \ l	A	B	C	D	E
1	1			−1	1
2		1			−1
3			1		−1
4				1	
5					1

c. The augmented network impedance matrix obtained by nonsingular transformation is

$$\hat{Z}_{LOOP} = \hat{C}^t[z]\hat{C} = \begin{array}{|c|c|} \hline Z_1 & Z_2 \\ \hline Z_3 & Z_4 \\ \hline \end{array}$$

=

	1	2	3	4	5
A	1				
B		1			
C			1		
D	−1			1	
E	1	−1	−1		1

	1	2	3	4	5
1	.6	.1		.2	
2	.1	.5			
3			.5		
4	.2			.4	
5					.2

	A	B	C	D	E
1	1			−1	1
2		1			−1
3			1		−1
4				1	
5					1

Problems

3.1 Select for the sample network shown in Fig. 3.10 a different tree than that used in the example. Retain node 1 as the reference and form:

a. The incidence matrices $\hat{A}$, A, K, B, $\hat{B}$, C, and $\hat{C}$ and verify the following relations:

i. $A_b K^t = U$

ii. $B_l = A_l K^t$

iii. $C_b = -B_l^t$

iv. $\hat{C}\hat{B}^t = U$

b. The network matrices Y_{BUS}, Y_{BR}, and Z_{LOOP} by singular transformations

c. The network matrices Z_{LOOP}, Z_{BR}, and Z_{BUS} by nonsingular transformations

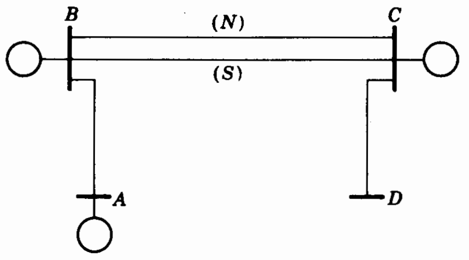


***Fig. 3.14** Sample power system for Prob. 3.2.*

3.2 The positive and zero sequence impedance data for the sample power system shown in Fig. 3.14 is given in Table 3.5. For this system:

a. Draw the positive sequence diagram and an oriented connected graph.

***Table 3.5** Positive and zero sequence impedance data of sample power system for Prob. 3.2*

Element	*Positive sequence impedance*	*Zero sequence impedance*	*Element*	*Mutual impedance*
Generator A	$0.0 + j0.25$	$0.0 + j0.1$		
Generator B	$0.0 + j0.25$	$0.0 + j0.1$		
Generator C	$0.0 + j0.25$	$0.0 + j0.1$		
Line A–B	$0.03 + j0.13$	$0.08 + j0.45$		
Line B–$C(N)$	$0.05 + j0.22$	$0.13 + j0.75$	Line B–$C(S)$	$0.08 + j0.48$
Line B–$C(S)$	$0.05 + j0.22$	$0.13 + j0.75$		
Line C–D	$0.02 + j0.11$	$0.07 + j0.37$		

c. The network matrices Y_{BUS}, Z_{BUS}, Z_{BR}, Z_{LOOP}, and Y_{LOOP} by nonsingular transformations

3.4 Prove that when there is no mutual coupling the diagonal and off-diagonal elements of the bus admittance matrix Y_{BUS} can be computed from

$$Y_{ii} = \sum_j y_{ij}$$

$$Y_{ij} = -y_{ij}$$

where y_{ij} is the sum of the admittances of all lines connecting buses i and j.

3.5 Using the bus impedance matrix Z_{BUS} computed in Prob. 3.2 and the internal generator voltages given in Table 3.7:

a. Compute the positive and zero sequence bus voltages of the network.

b. Compute the positive and zero sequence currents flowing in the line B–$C(N)$.

Table 3.7 Internal generator voltages for Prob. 3.5

	Internal per unit voltages	
Generator	*Positive sequence*	*Zero sequence*
A	$1.0/0°$	0
B	$1.1/-10°$	0
C	$1.0/-10°$	$0.1/0°$

3.6 Using the relations between interconnected and primitive network variables prove the following:

a. $A_b K^t = U$

b. $B_l = A_l K^t$

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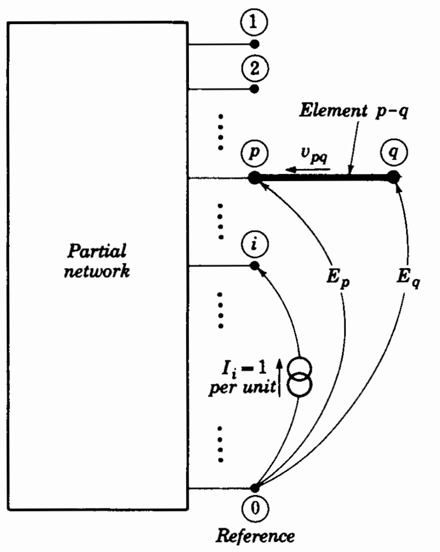

***Fig. 4.3** Injected current and bus voltages for calculation of* Z_{qi}.

node as shown in Fig. 4.3. Since all other bus currents equal zero, it follows from equation (4.2.1) that

$$
\begin{aligned}
E_1 &= Z_{1i}I_i \\
E_2 &= Z_{2i}I_i \\
&\cdots\cdots \\
E_p &= Z_{pi}I_i \\
&\cdots\cdots \\
E_m &= Z_{mi}I_i \\
E_q &= Z_{qi}I_i
\end{aligned}
\qquad (4.2.2)
$$

Letting $I_i = 1$ per unit in equations (4.2.2), Z_{qi} can be obtained directly by calculating E_q.

The bus voltages associated with the added element and the voltage across the element are related by

$$E_q = E_p - v_{pq} \qquad (4.2.3)$$

The currents in the elements of the network in Fig. 4.3 are expressed in terms of the primitive admittances and the voltages across the elements by

$$
\begin{bmatrix} i_{pq} \\ i_{\rho\sigma} \end{bmatrix} = \begin{bmatrix} y_{pq,pq} & y_{pq,\rho\sigma} \\ y_{\rho\sigma,pq} & y_{\rho\sigma,\rho\sigma} \end{bmatrix} \begin{bmatrix} v_{pq} \\ v_{\rho\sigma} \end{bmatrix} \qquad (4.2.4)
$$

where i_{ij} = current through the element i–j

e_{pq} = voltage source in series with the element p–q

Let $e_{pq} = 1$ per unit, as shown in Fig. 4.7, then

$$Y_{ij,pq} = i_{ij} \tag{4.5.4}$$

It remains therefore to calculate the current i_{ij}.

The performance equation in admittance form for the primitive network is

$$\bar{\imath} + \bar{\jmath} = [y]\bar{v}$$

The current through the element i–j is

$$i_{ij} = -j_{ij} + \bar{y}_{ij,\rho\sigma}\bar{v}_{\rho\sigma} \tag{4.5.5}$$

where $\rho\sigma$ refers to all elements of the network. The voltage source in series with p–q induces currents in the elements mutually coupled with p–q. This voltage source can be replaced by equivalent current sources in parallel with each element, as shown in Fig. 4.8. The equivalent

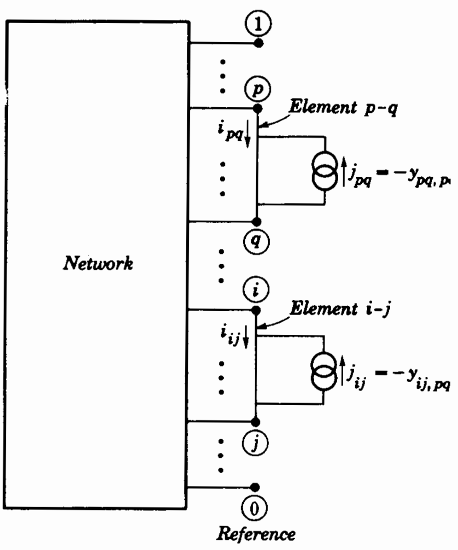

Fig. 4.8 Equivalent source currents for calculation of $Y_{ij,pq}$.

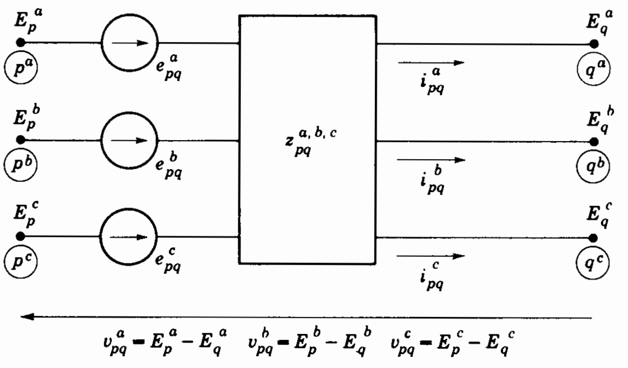

Fig. 5.1 **Representation of three-phase network component in impedance form.**

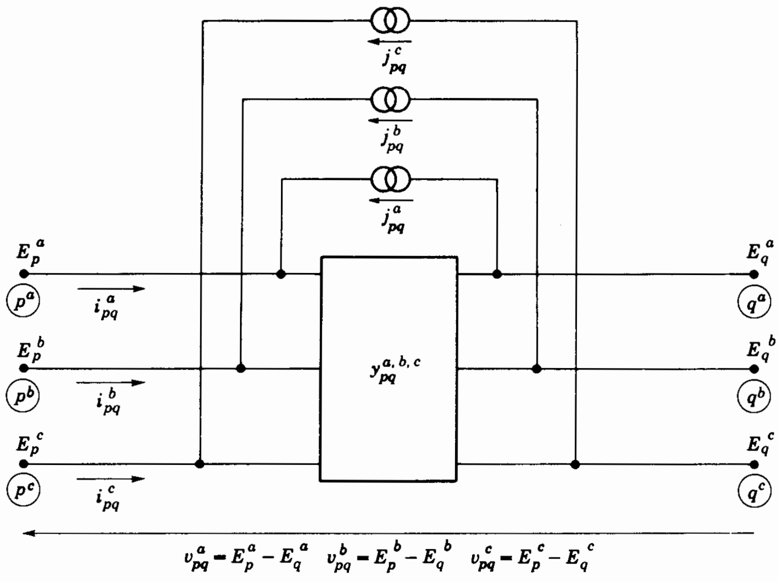

Fig. 5.2 **Representation of three-phase network component in admittance form.**

$i^a_{pq}, i^b_{pq}, i^c_{pq}$ are the currents through the element p–q for phases a, b, and c, respectively

$j^a_{pq}, j^b_{pq}, j^c_{pq}$ are the source currents in parallel with phases a, b, and c, respectively, of the element p–q

$z^{a,b,c}_{pq}$ is the three-phase impedance matrix for the element p–q

$y^{a,b,c}_{pq}$ is the three-phase admittance matrix for the element p–q

The performance equation of a three-phase element in impedance form is

$$\begin{bmatrix} v^a_{pq} \\ v^b_{pq} \\ v^c_{pq} \end{bmatrix} + \begin{bmatrix} e^a_{pq} \\ e^b_{pq} \\ e^c_{pq} \end{bmatrix} = \begin{bmatrix} z^{aa}_{pq} & z^{ab}_{pq} & z^{ac}_{pq} \\ z^{ba}_{pq} & z^{bb}_{pq} & z^{bc}_{pq} \\ z^{ca}_{pq} & z^{cb}_{pq} & z^{cc}_{pq} \end{bmatrix} \begin{bmatrix} i^a_{pq} \\ i^b_{pq} \\ i^c_{pq} \end{bmatrix} \tag{5.2.1}$$

where z^{aa}_{pq} = self-impedance of phase a of the three-phase element connecting nodes p and q

z^{ab}_{pq} = mutual impedance between phases a and b

z^{ac}_{pq} = mutual impedance between phases a and c

and so forth.

Equation (5.2.1) can be written more concisely as

$$v^{a,b,c}_{pq} + e^{a,b,c}_{pq} = z^{a,b,c}_{pq} i^{a,b,c}_{pq} \tag{5.2.2}$$

The performance equation in admittance form is

$$\begin{bmatrix} i^a_{pq} \\ i^b_{pq} \\ i^c_{pq} \end{bmatrix} + \begin{bmatrix} j^a_{pq} \\ j^b_{pq} \\ j^c_{pq} \end{bmatrix} = \begin{bmatrix} y^{aa}_{pq} & y^{ab}_{pq} & y^{ac}_{pq} \\ y^{ba}_{pq} & y^{bb}_{pq} & y^{bc}_{pq} \\ y^{ca}_{pq} & y^{cb}_{pq} & y^{cc}_{pq} \end{bmatrix} \begin{bmatrix} v^a_{pq} \\ v^b_{pq} \\ v^c_{pq} \end{bmatrix}$$

which can be written

$$i^{a,b,c}_{pq} + j^{a,b,c}_{pq} = y^{a,b,c}_{pq} v^{a,b,c}_{pq}$$

where

$$y^{a,b,c}_{pq} = (z^{a,b,c}_{pq})^{-1}$$

The parallel three-phase source current in admittance form and the three-phase series source voltage in impedance form have the relationship, as is the case in single-phase representation,

$$j^{a,b,c}_{pq} = -y^{a,b,c}_{pq} e^{a,b,c}_{pq}$$

The impedance matrix $z_{pq}^{a,b,c}$ and the admittance matrix $y_{pq}^{a,b,c}$ of a stationary bilateral element are symmetric. If, in addition, the three-phase element is balanced, then the diagonal elements of $z_{pq}^{a,b,c}$, designated by z_{pq}^{s}, are equal and the off-diagonal elements, designated by z_{pq}^{m}, are equal, that is,

$$z_{pq}^{aa} = z_{pq}^{bb} = z_{pq}^{cc} = z_{pq}^{s}$$

and

$$z_{pq}^{ab} = z_{pq}^{ac} = z_{pq}^{ba} = z_{pq}^{bc} = z_{pq}^{ca} = z_{pq}^{cb} = z_{pq}^{m}$$

The corresponding relations are true in the admittance matrix $y_{pq}^{a,b,c}$.

The impedance and admittance matrices of balanced three-phase rotating elements are not symmetric. However, the mutual coupling from phase a to phase b, b to c, and c to a for the phase sequence a, b, c are identical, that is,

$$z_{pq}^{ab} = z_{pq}^{bc} = z_{pq}^{ca} = z_{pq}^{m1}$$

Similarly,

$$z_{pq}^{ac} = z_{pq}^{ba} = z_{pq}^{cb} = z_{pq}^{m2}$$

The performance equation of the three-phase primitive network in impedance form is

$$\bar{v}^{a,b,c} + \bar{e}^{a,b,c} = [z^{a,b,c}]\bar{\imath}^{a,b,c}$$

or in the admittance form is

$$\bar{\imath}^{a,b,c} + \bar{\jmath}^{a,b,c} = [y^{a,b,c}]\bar{v}^{a,b,c}$$

The vectors representing the variables are composed of 3×1 submatrices corresponding to the variables of a particular three-phase network element. The parameter matrices are composed of 3×3 submatrices. These submatrices correspond to the self and mutual three-phase impedance or admittance matrices of the network elements.

5.3 Three-phase balanced network elements

Balanced excitation

The excitation of any three-phase element is balanced when the source voltages or source currents of all phases are equal in magnitude and displaced from each other by 120°. For balanced excitation,

$$e_{pq}^{a,b,c} = \begin{bmatrix} e_{pq}^{a} \\ e_{pq}^{b} \\ e_{pq}^{c} \end{bmatrix} = \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} e_{pq}^{a} \quad \text{and} \quad j_{pq}^{a,b,c} = \begin{bmatrix} j_{pq}^{a} \\ j_{pq}^{b} \\ j_{pq}^{c} \end{bmatrix} = \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} j_{pq}^{a}$$

where

$$a = \epsilon^{j(2\pi/3)} = -\tfrac{1}{2} + j\tfrac{1}{2}\sqrt{3}$$

It follows that $a^3 = 1, a^2 + a + 1 = 0$ and $a^* = a^2$. The phase voltages and phase currents are balanced if the excitation of a balanced three-phase element is balanced. Then, the performance equation, in impedance form for a stationary element is

$$\begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} v_{pq}^{a} + \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} e_{pq}^{a} = \begin{bmatrix} z_{pq}^{s} & z_{pq}^{m} & z_{pq}^{m} \\ z_{pq}^{m} & z_{pq}^{s} & z_{pq}^{m} \\ z_{pq}^{m} & z_{pq}^{m} & z_{pq}^{s} \end{bmatrix} \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} i_{pq}^{a} \qquad (5.3.1)$$

and for a rotating element is

$$\begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} v_{pq}^{a} + \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} e_{pq}^{a} = \begin{bmatrix} z_{pq}^{s} & z_{pq}^{m1} & z_{pq}^{m2} \\ z_{pq}^{m2} & z_{pq}^{s} & z_{pq}^{m1} \\ z_{pq}^{m1} & z_{pq}^{m2} & z_{pq}^{s} \end{bmatrix} \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} i_{pq}^{a} \qquad (5.3.2)$$

Both sides of equation (5.3.1) can be premultiplied by the conjugate transpose of

$$\begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix}$$

The augmented impedance matrix is

		② a	② b	② c	③ a	③ b	③ c	④ a	④ b	④ c	l a	l b	l c
	a	.0740	−.0219	−.0176	.0060	−.0031	−.0024	.0468	−.0144	−.0115	.0612	−.0256	−.0220
②	b	−.0176	.0740	−.0219	−.0024	.0060	−.0031	−.0115	.0468	−.0144	−.0220	.0612	−.0256
	c	−.0219	−.0176	.0740	−.0031	−.0024	.0060	−.0144	−.0115	.0468	−.0256	−.0220	.0612
	a	.0060	−.0031	−.0024	.0740	−.0219	−.0176	.0332	−.0106	−.0085	−.0612	.0256	.0220
③	b	−.0024	.0060	−.0031	−.0176	.0740	−.0219	−.0085	.0332	−.0106	.0220	−.0612	.0256
	c	−.0031	−.0024	.0060	−.0219	−.0176	.0740	−.0106	−.0085	.0332	.0256	.0220	−.0612
	a	.0468	−.0144	−.0115	.0332	−.0106	−.0085	.4014	.1071	.1097	.0122	−.0052	−.0044
④	b	−.0115	.0468	−.0144	−.0085	.0332	−.0106	.1097	.4014	.1071	−.0044	.0122	−.0052
	c	−.0144	−.0115	.0468	−.0106	−.0085	.0332	.1071	.1097	.4014	−.0052	−.0044	.0122
	a	.0612	−.0256	−.0220	−.0612	.0256	.0220	.0122	−.0052	−.0044	1.3170	.1434	.1506
l	b	−.0220	.0612	−.0256	.0220	−.0612	.0256	−.0044	.0122	−.0052	.1506	1.3170	.1434
	c	−.0256	−.0220	.0612	.0256	.0220	−.0612	−.0052	−.0044	.0122	.1434	.1506	1.3170

Eliminating the rows and columns corresponding to the fictitious node l, the bus impedance matrix of the system is

$Z_{BUS}^{a,b,c} =$

		② a	② b	② c	③ a	③ b	③ c	④ a	④ b	④ c
②	a	.0699	−.0196	−.0159	.0101	−.0054	−.0041	.0460	−.0139	−.0112
②	b	−.0159	.0699	−.0196	−.0041	.0101	−.0054	−.0112	.0460	−.0139
②	c	−.0196	−.0159	.0699	−.0054	−.0041	.0101	−.0139	−.0112	.0460
③	a	.0101	−.0054	−.0041	.0699	−.0196	−.0159	.0340	−.0111	−.0088
③	b	−.0041	.0101	−.0054	−.0159	.0699	−.0196	−.0088	.0340	−.0111
③	c	−.0054	−.0041	.0101	−.0196	−.0159	.0699	−.0111	−.0088	.0340
④	a	.0460	−.0139	−.0112	.0340	−.0111	−.0088	.4012	.1072	.1098
④	b	−.0112	.0460	−.0139	−.0088	.0340	−.0111	.1098	.4012	.1072
④	c	−.0139	−.0112	.0460	−.0111	−.0088	.0340	.1072	.1098	.4012

This matrix can be checked by inverting the bus admittance matrix obtained in part b of this problem.

The new primitive admittance submatrix is

$[y_s'^{a,b,c}]$ =

		2–4 a	2–4 b	2–4 c	4–3 a	4–3 b	4–3 c	2–3 a	2–3 b	2–3 c
2–4	a	2.0000	−.5000	−.5000						
	b	−.5000	2.0000	−.5000						
	c	−.5000	−.5000	2.0000						
4–3	a				1.3333	−.3333	−.3333			
	b				−.3333	1.3333	−.3333			
	c				−.3333	−.3333	1.3333			
2–3	a									
	b									
	c									

Substituting, then

$[M^{a,b,c}] =$

1.00135	.00135	.00135	.00203	.00203	.00203	.00338	.00338	.00338
.00135	1.00135	.00135	.00203	.00203	.00203	.00338	.00338	.00338
.00135	.00135	1.00135	.00203	.00203	.00203	.00338	.00338	.00338
.00135	.00135	.00135	1.00203	.00203	.00203	.00338	.00338	.00338
.00135	.00135	.00135	.00203	1.00203	.00203	.00338	.00338	.00338
.00135	.00135	.00135	.00203	.00203	1.00203	.00338	.00338	.00338
−.04105	.01815	.01615	−.06165	.02715	.02435	.89730	.04530	.04050
.01615	−.04105	.01815	.02435	−.06165	.02715	.04050	.89730	.04530
.01815	.01615	−.04105	.02715	.02435	−.06165	.04530	.04050	.89730

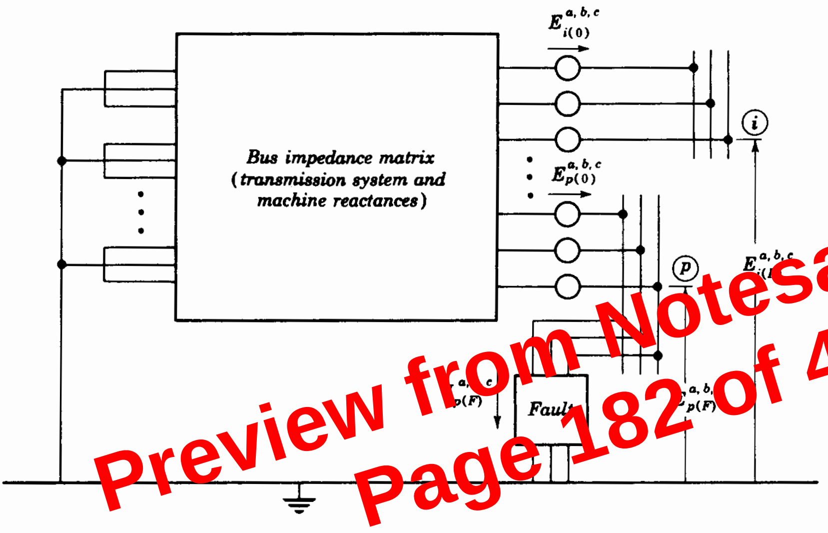

Fig. 6.3 Three-phase representation of a power system with a fault at bus p.

where the elements of $Z^{a,b,c}_{BUS}$ are matrices of dimension 3 × 3. Equation (6.2.1) can be written as follows:

$$
\begin{aligned}
E^{a,b,c}_{1(F)} &= E^{a,b,c}_{1(0)} - Z^{a,b,c}_{1p} I^{a,b,c}_{p(F)} \\
E^{a,b,c}_{2(F)} &= E^{a,b,c}_{2(0)} - Z^{a,b,c}_{2p} I^{a,b,c}_{p(F)} \\
&\cdots \\
E^{a,b,c}_{p(F)} &= E^{a,b,c}_{p(0)} - Z^{a,b,c}_{pp} I^{a,b,c}_{p(F)} \\
&\cdots \\
E^{a,b,c}_{n(F)} &= E^{a,b,c}_{n(0)} - Z^{a,b,c}_{np} I^{a,b,c}_{p(F)}
\end{aligned}
\tag{6.2.2}
$$

The three-phase voltage vector at the faulted bus p is, from Fig. 6.3,

$$E^{a,b,c}_{p(F)} = Z^{a,b,c}_{F} I^{a,b,c}_{p(F)} \tag{6.2.3}$$

where $Z^{a,b,c}_{F}$ is the three-phase impedance matrix for the fault. The elements of this 3 × 3 matrix depend on the type of fault and fault impedance. Substituting from equation (6.2.3) for $E^{a,b,c}_{p(F)}$, the pth equation of (6.2.2) becomes

$$Z^{a,b,c}_{F} I^{a,b,c}_{p(F)} = E^{a,b,c}_{p(0)} - Z^{a,b,c}_{pp} I^{a,b,c}_{p(F)} \tag{6.2.4}$$

Solving equation (6.2.4) for $I^{a,b,c}_{p(F)}$ yields

$$I^{a,b,c}_{p(F)} = (Z^{a,b,c}_{F} + Z^{a,b,c}_{pp})^{-1} E^{a,b,c}_{p(0)} \tag{6.2.5}$$

Substituting for $I^{a,b,c}_{p(F)}$ in equation (6.2.3), the three-phase voltage at the faulted bus p is

$$E^{a,b,c}_{p(F)} = Z^{a,b,c}_{F}(Z^{a,b,c}_{F} + Z^{a,b,c}_{pp})^{-1}E^{a,b,c}_{p(0)} \qquad (6.2.6)$$

Similarly, the three-phase voltages at buses other than p can be obtained by substituting for $I^{a,b,c}_{p(F)}$ from equation (6.2.5). Then

$$E^{a,b,c}_{i(F)} = E^{a,b,c}_{i(0)} - Z^{a,b,c}_{ip}(Z^{a,b,c}_{F} + Z^{a,b,c}_{pp})^{-1}E^{a,b,c}_{p(0)} \qquad i \neq p \qquad (6.2.7)$$

When it is desirable to express the parameters of the fault circuit in the admittance form, the three-phase fault current at bus p is

$$I^{a,b,c}_{p(F)} = Y^{a,b,c}_{F}E^{a,b,c}_{p(F)} \qquad (6.2.8)$$

where $Y^{a,b,c}_{F}$ is the three-phase admittance matrix for the fault. Substituting $I^{a,b,c}_{p(F)}$ from equation (6.2.8), the pth equation of (6.2.2) becomes

$$E^{a,b,c}_{p(F)} = E^{a,b,c}_{p(0)} - Z^{a,b,c}_{pp}Y^{a,b,c}_{F}E^{a,b,c}_{p(F)} \qquad (6.2.9)$$

Solving equation (6.2.9) for $E^{a,b,c}_{p(F)}$ yields

$$E^{a,b,c}_{p(F)} = (U + Z^{a,b,c}_{pp}Y^{a,b,c}_{F})^{-1}E^{a,b,c}_{p(0)} \qquad (6.2.10)$$

Substituting for $E^{a,b,c}_{p(F)}$ in equation (6.2.8), the three-phase current at the faulted bus p is

$$I^{a,b,c}_{p(F)} = Y^{a,b,c}_{F}(U + Z^{a,b,c}_{pp}Y^{a,b,c}_{F})^{-1}E^{a,b,c}_{p(0)} \qquad (6.2.11)$$

Similarly, the three-phase voltages at buses other than p can be obtained by substituting for $I^{a,b,c}_{p(F)}$ from equation (6.2.11). Then

$$E^{a,b,c}_{i(F)} = E^{a,b,c}_{i(0)} - Z^{a,b,c}_{ip}Y^{a,b,c}_{F}(U + Z^{a,b,c}_{pp}Y^{a,b,c}_{F})^{-1}E^{a,b,c}_{p(0)} \qquad i \neq p \qquad (6.2.12)$$

Fault currents flowing through the elements of the network can be calculated with the bus voltages obtained from equations (6.2.6) and (6.2.7) or from equations (6.2.10) and (6.2.12). These currents in terms of the voltages across the elements of the network are

$$\bar{i}^{a,b,c}_{(F)} = [y^{a,b,c}]\bar{v}^{a,b,c}_{(F)}$$

where the elements of the current vector are

$$i^{a,b,c}_{ij(F)} = \begin{bmatrix} i^{a}_{ij(F)} \\ i^{b}_{ij(F)} \\ i^{c}_{ij(F)} \end{bmatrix}$$

Since positive and negative primitive sequence impedances are equal, the positive and negative sequence bus impedance matrices are equal.

The procedure for forming the zero sequence bus impedance matrix is identical for the first four steps. The zero sequence bus impedance matrix of the partial network, before adding element 3, is

$Z^{(0)}_{BUS} =$

	②	③	④
②	0.0345	0.0005	0.0209
③	0.0005	0.0345	0.0141
④	0.0209	0.0141	0.6182

Step 5. Add element 3, the link from $p = 2$ to $q = 3$, which is coupled with the elements 4 and 5. The elements of the row and column corresponding to the fictitious node l are

$$Z_{l2} = Z_{22} - Z_{32} + \frac{y_{23,24}(Z_{22} - Z_{42}) + y_{23,43}(Z_{42} - Z_{32})}{y_{23,23}}$$

$$Z_{l3} = Z_{23} - Z_{33} + \frac{y_{23,24}(Z_{23} - Z_{43}) + y_{23,43}(Z_{43} - Z_{33})}{y_{23,23}}$$

$$Z_{l4} = Z_{24} - Z_{34} + \frac{y_{23,24}(Z_{24} - Z_{44}) + y_{23,43}(Z_{44} - Z_{34})}{y_{23,23}}$$

$$Z_{ll} = Z_{2l} - Z_{3l} + \frac{1 + y_{23,24}(Z_{2l} - Z_{4l}) + y_{23,43}(Z_{4l} - Z_{3l})}{y_{23,23}}$$

The zero sequence primitive impedance matrix is

	1–2	1–3	2–3	2–4	4–3
1–2	0.035				
1–3		0.035			
2–3			2.500	0.600	0.900
2–4			0.600	1.000	
4–3			0.900		1.500

Combining the elements of the three sequence impedance matrices, the bus impedance matrix is

$Z_{BUS}^{0,1,2} =$

		② 0	② 1	② 2	③ 0	③ 1	③ 2	④ 0	④ 1	④ 2
②	0	0.0344			0.0006			0.0209		
②	1		0.0876			0.0149			0.0586	
②	2			0.0876			0.0149			0.0586
③	0	0.0006			0.0344			0.0141		
③	1		0.0149			0.0876			0.0439	
③	2			[illegible]			0.0876			0.0439
④	0	0.0209			0.014[illegible]			0.6182		
④	1		0.0586			0.0439			0.2928	
④	2			0.0586			0.0439			0.2928

Assuming the fault impedance is zero, the total fault current for a three-phase fault at bus 4 is

$$I_{4(F)}^{0,1,2} = \begin{bmatrix} 0 \\ \dfrac{\sqrt{3}}{Z_{44}^{(1)} + z_F} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{\sqrt{3}}{0.2928} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3.42\sqrt{3} \\ 0 \end{bmatrix}$$

The phase components of the fault current are

$$I_{4(F)}^{a,b,c} = T_s I_{4(F)}^{0,1,2} = 3.42 \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix}$$

b. The fault current for a line-to-ground fault at bus 4, assuming zero fault impedance, is

$$I_{4(F)}^{0,1,2} = \frac{\sqrt{3}}{Z_{44}^{(0)} + 2Z_{44}^{(1)} + 3z_F}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{\sqrt{3}}{0.6182 + 0.5856}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}0.83\sqrt{3}\\0.83\sqrt{3}\\0.83\sqrt{3}\end{bmatrix}$$

The phase components of the total fault current are

$$I_{4(F)}^{a,b,c} = T_s I_{4(F)}^{0,1,2} = \begin{bmatrix}2.49\\0\\0\end{bmatrix}$$

Bus voltages during the fault are

$$E_{4(F)}^{0,1,2} = \frac{\sqrt{3}}{Z_{44}^{(0)} + 2Z_{44}^{(1)} + 3z_F}\begin{bmatrix}-Z_{44}^{(0)}\\Z_{44}^{(0)} + Z_{44}^{(1)}\\-Z_{44}^{(1)}\end{bmatrix} = 0.83\sqrt{3}\begin{bmatrix}-0.6182\\0.9110\\-0.2928\end{bmatrix}$$

$$= \begin{bmatrix}-0.5131\sqrt{3}\\0.7561\sqrt{3}\\-0.2430\sqrt{3}\end{bmatrix}$$

$$E_{2(F)}^{0,1,2} = \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} - \frac{\sqrt{3}}{Z_{44}^{(0)} + 2Z_{44}^{(1)} + 3z_F} \begin{bmatrix} Z_{42}^{(0)} \\ Z_{42}^{(1)} \\ Z_{42}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} - 0.83\sqrt{3} \begin{bmatrix} 0.0209 \\ 0.0586 \\ 0.0586 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0173\sqrt{3} \\ 0.9514\sqrt{3} \\ -0.0486\sqrt{3} \end{bmatrix}$$

$$E_{3(F)}^{0,1,2} = \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} - \frac{\sqrt{3}}{Z_{44}^{(0)} + 2Z_{44}^{(1)} + 3z_F} \begin{bmatrix} Z_{43}^{(0)} \\ Z_{43}^{(1)} \\ Z_{43}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} - 0.83\sqrt{3} \begin{bmatrix} 0.0141 \\ 0.0439 \\ 0.0439 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0117\sqrt{3} \\ 0.9636\sqrt{3} \\ -0.0364\sqrt{3} \end{bmatrix}$$

$$
i^{0,1,2}_{24(F)} = \begin{bmatrix} y^{(0)}_{24,24}(E^{(0)}_{2(F)} - E^{(0)}_{4(F)}) + y^{(0)}_{24,43}(E^{(0)}_{4(F)} - E^{(0)}_{3(F)}) + y^{(0)}_{24,23}(E^{(0)}_{2(F)} - E^{(0)}_{3(F)}) \\ y^{(1)}_{24,24}(E^{(1)}_{2(F)} - E^{(1)}_{4(F)}) \\ y^{(2)}_{24,24}(E^{(2)}_{2(F)} - E^{(2)}_{4(F)}) \end{bmatrix}
$$

$$
= \sqrt{3} \begin{bmatrix} 1.225(-0.0173 + 0.5131) + 0.225(-0.5131 + 0.0117) + (-0.375)(-0.0173 + 0.0117) \\ \frac{1}{0.4}(0.9514 - 0.7561) \\ \frac{1}{0.4}(-0.0486 + 0.2430) \end{bmatrix}
$$

$$
= \begin{bmatrix} 0.50\sqrt{3} \\ 0.49\sqrt{3} \\ 0.49\sqrt{3} \end{bmatrix}
$$

The phase components of the currents in the lines connected to the fault bus are

$$
i^{a,b,c}_{43(F)} = T_s i^{0,1,2}_{43(F)} = \begin{bmatrix} -1.02 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad i^{a,b,c}_{24(F)} = T_s i^{0,1,2}_{24(F)} = \begin{bmatrix} 1.47 \\ 0 \\ 0 \end{bmatrix}
$$

c. The fault currents occurring for a fault on the line side of breaker A can be calculated by assuming the fault on bus 4, since both locations are electrically equivalent. When this type of fault occurs and breaker A opens before breaker B, the interrupted current will be the total fault current at bus 4 less the fault contribution flowing from bus 3 over line 5.

fault since all bus currents and off-nominal tap settings are neglected. It is necessary, therefore, to calculate the loop currents resulting from the fault in order to determine short circuit currents and voltages. The fault calculations can be performed using either three-phase quantities or symmetrical components. The method will be described using three-phase quantities.

The number of three-phase elements in the simplified system is equal to the number of network elements plus the number of machine equivalents. The number of nodes is equal to the number of buses n plus ground, that is $n + 1$. The number of links or basic loops, in the simplified system, is, then

$$l_n = (e + e_g) - (n + 1) + 1$$

or

$$l_n = e + e_g - n$$

where e is the number of three-phase network elements and e_g is the number of three-phase machine equivalents.

A fault at bus p is simulated by adding a link from the bus to ground. Using the representation of the system shown in Fig. 6.3, the voltages during the fault are

$$\bar{E}^{a,b,c}_{BUS(F)} = \bar{E}^{a,b,c}_{BUS(0)} + \overline{\Delta E}^{a,b,c}_{BUS} \tag{6.5.1}$$

where the vector $\overline{\Delta E}^{a,b,c}_{BUS}$ represents changes in bus voltages resulting from the faulted bus source voltage $E^{a,b,c}_{p(0)}$.

The performance equation of a network in the loop frame of reference is

$$\bar{E}^{a,b,c}_{LOOP} = Z^{a,b,c}_{LOOP}\bar{I}^{a,b,c}_{LOOP}$$

For the faulted system, shown in Fig. 6.3, the known loop voltage vector is

$$\bar{E}^{a,b,c}_{LOOP} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ E^{a,b,c}_{p(0)} \end{bmatrix}$$

The dimension of the loop impedance matrix, which includes the fault loop, is $3(l_n + 1) \times 3(l_n + 1)$. The unknown loop current vector due loops, can be partitioned as follows:

$$\begin{array}{|c|}\hline E_L^{a,b,c} \\ \hline E_{L(F)}^{a,b,c} \\ \hline \end{array} = \begin{array}{|c|c|}\hline Z_L^{a,b,c} & Z_M^{a,b,c} \\ \hline (Z_M^{a,b,c})^t & Z_A^{a,b,c} \\ \hline \end{array} \begin{array}{|c|}\hline I_L^{a,b,c} \\ \hline I_{L(F)}^{a,b,c} \\ \hline \end{array} \qquad (6.5.4)$$

In equation (6.5.4), the vectors $\bar{E}_L^{a,b,c}$ and $\bar{I}_L^{a,b,c}$ refer to the loops in the simplified system and $\bar{E}_{L(F)}^{a,b,c}$ and $\bar{I}_{L(F)}^{a,b,c}$ refer to the auxiliary loops.

The vector $\bar{I}_L^{a,b,c}$ can be calculated for a fault at bus p from equation (6.5.4) by assuming the auxiliary loop currents to be

$$\bar{I}_{L(F)}^{a,b,c} = \begin{array}{|c|}\hline 0 \\ \hline \cdots \\ \hline 0 \\ \hline I_{Lp(F)}^{a,b,c} \\ \hline 0 \\ \hline \cdots \\ \hline 0 \\ \hline \end{array} \qquad (6.5.5)$$

where $I_{Lp(F)}^{a,b,c}$ is the assumed three-phase current vector of the pth auxiliary loop. From equation (6.5.4) it follows that

$$Z_L^{a,b,c}\bar{I}_L^{a,b,c} + Z_M^{a,b,c}\bar{I}_{L(F)}^{a,b,c} = \bar{E}_L^{a,b,c} \qquad (6.5.6)$$

Since $\bar{E}_L^{a,b,c} = 0$, equation (6.5.6) becomes

$$Z_L^{a,b,c}\bar{I}_L^{a,b,c} + Z_M^{a,b,c}\bar{I}_{L(F)}^{a,b,c} = 0$$

Solving for the loop currents of the simplified system,

$$\bar{I}_L^{a,b,c} = -(Z_L^{a,b,c})^{-1}Z_M^{a,b,c}\bar{I}_{L(F)}^{a,b,c} \qquad (6.5.7)$$

c. The ground current at bus B for a fault with the reactance

$$x_{0B} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & \infty \\ \hline \end{array}$$

d. The fault voltages at buses A and B

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Except for the last multiplications to obtain the value of the determinant, this process is identical to that performed in the forward course of the elimination method. In accordance with Cramer's rule the solution of a linear system is found by

$$x_i = \frac{|A_i|}{|A|} \qquad i = 1, 2, \ldots, n$$

and requires the evaluation of $n + 1$ determinants of order n. Thus the computation required for the Gauss elimination method to obtain a complete solution only slightly exceeds that required to evaluate a single determinant. This shows the inefficiency of the use of Cramer's rule for the solution of linear sets of equations.

Solution of multiple sets of equations and matrix inversion

The Gauss elimination method may be applied to the simultaneous solution of several sets of linear equations for which only the known constants differ. This is accomplished by adjoining all the constant vectors $y_i^{(1)}$, $y_i^{(2)}, \ldots, y_i^{(n)}$ to the coefficient matrix as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{bmatrix}$$

The elimination process is then applied to the entire array. Each back substitution must be performed separately for the Gauss elimination method, but for the Gauss-Jordan method all solutions are obtained directly.

If many constant vectors are given, it may be advantageous to obtain the inverse of the coefficient matrix and then multiply this inverse by each of the constant vectors in turn to obtain each solution. Given the set of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= y_1^{(1)} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= y_2^{(1)} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= y_3^{(1)} \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= y_4^{(1)} \end{aligned} \tag{7.2.13}$$

the solution is, from equation (7.2.3),

$$\begin{aligned} x_1 &= b_{11}y_1^{(1)} + b_{12}y_2^{(1)} + b_{13}y_3^{(1)} + b_{14}y_4^{(1)} \\ x_2 &= b_{21}y_1^{(1)} + b_{22}y_2^{(1)} + b_{23}y_3^{(1)} + b_{24}y_4^{(1)} \\ x_3 &= b_{31}y_1^{(1)} + b_{32}y_2^{(1)} + b_{33}y_3^{(1)} + b_{34}y_4^{(1)} \\ x_4 &= b_{41}y_1^{(1)} + b_{42}y_2^{(1)} + b_{43}y_3^{(1)} + b_{44}y_4^{(1)} \end{aligned} \tag{7.2.14}$$

Solution

The data for the network is given in Table 7.1. The impedance of the generator is 0.01 and the voltage behind the generator is assumed equal to one per unit. The loop equations of the network are

$$1.0 = 0.01(I_1 + I_2 + I_3) + (0.3380 + 0.2790)I_1 + 0.1830I_2$$
$$1.0 = 0.01(I_1 + I_2 + I_3) + 0.4740I_2 + (0.0251 + 0.1360)I_3 + 0.1830I_1$$
$$1.0 = 0.01(I_1 + I_2 + I_3) + (0.5000 + 0.1860)I_3 + (0.0251 + 0.1360)I_2$$

Combining terms

$$0.6270I_1 + 0.1930I_2 + 0.0100I_3 = 1.0$$
$$0.1930I_1 + 0.4840I_2 + 0.1711I_3 = 1.0$$
$$0.0100I_1 + 0.1711I_2 + 0.6960I_3 = 1.0$$

a. The forward course in the Gauss elimination method for the solution of the linear equations is shown in Table 7.2. The original coefficients of the matrix and constant terms are given in part *a*. Included also is the control sum obtained by adding the coefficients and constant term of each row. If the same operations are performed on this sum as on the coefficients and constant term, the control sum will equal at each stage the sum of the elements of the row. This provides a check on the arithmetic operations of the process.

The process is initiated by dividing all elements in the first row by 0.6270, the leading coefficient. The resulting elements are given in the first row of part *b* of Table 7.2. The elements of this new row are multiplied then by 0.1930, the leading coefficient of the second row. The resulting terms are subtracted from the elements of the second row to obtain a new second row as shown in part *b*. Next, the elements of the first row are multiplied by 0.0100 and the resultant terms are subtracted from the elements of the third row. This procedure is repeated for the second and third rows by first dividing the elements of the second row by

Table 7.1 Impedance data for sample system

Self		*Mutual*	
Bus code $p-q$	*Impedance* $z_{pq,pq}$	*Bus code* $r-s$	*Impedance* $z_{pq,rs}$
1–2	0.5000	1–3	0.0251
1–3	0.4740	2–3	0.1360
1–4	0.3380	1–3	0.1830
2–3	0.1860		
3–4	0.2790		

Gauss and Gauss-Seidel iterative methods

Explicitly, the iterative process described requires the selection of initial values

$$x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$$

to use in the formulas

$$x_1^{k+1} = \frac{y_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2{}^k - \frac{a_{13}}{a_{11}} x_3{}^k \cdots - \frac{a_{1n}}{a_{11}} x_n{}^k$$

$$x_2^{k+1} = \frac{y_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1{}^k - \frac{a_{23}}{a_{22}} x_3{}^k \cdots - \frac{a_{2n}}{a_{22}} x_n{}^k$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n^{k+1} = \frac{y_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1{}^k - \frac{a_{n2}}{a_{nn}} x_2{}^k \cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^k$$

in order to calculate a second estimate

$$x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$$

The superscript k refers to the iteration count. These calculated values are used in the formulas to obtain a third estimate

$$x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$$

and so forth. This technique is referred to as the Gauss iterative method in which the new estimates are substituted only after all equations have been processed.

An alternative method is to make an immediate substitution in subsequent equations for each new value of x as it is obtained. Thus, in the solution of x_i the values used would be

$$x_1^{k+1}, x_2^{k+1}, \ldots, x_{i-1}^{k+1}, x_{i+1}^k, \ldots, x_n{}^k$$

This iteration scheme, called Gauss-Seidel, has the following formulas:

$$x_1^{k+1} = \frac{y_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2{}^k \cdots\cdots\cdots\cdots - \frac{a_{1n}}{a_{11}} x_n{}^k$$

$$x_2^{k+1} = \frac{y_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{k+1} \cdots\cdots\cdots\cdots - \frac{a_{2n}}{a_{22}} x_n{}^k$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_i^{k+1} = \frac{y_i}{a_{ii}} - \frac{a_{i1}}{a_{ii}} x_1^{k+1} \cdots - \frac{a_{i,i-1}}{a_{ii}} x_{i-1}^{k+1} - \frac{a_{i,i+1}}{a_{ii}} x_{i+1}^k \cdots - \frac{a_{in}}{a_{ii}} x_n{}^k$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n^{k+1} = \frac{y_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{k+1} \cdots\cdots\cdots\cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{k+1} \tag{7.4.3}$$

is continued until the values of all correction terms are less than a specified tolerance. Final values of the variables x_i are obtained from the equation

$$x_i = x_i^{(0)} + \sum_k \alpha_i^k$$

Relaxation method

The methods of Gauss and Gauss-Seidel are used to solve linear algebraic equations by successive approximations or corrections. These methods treat the equations in the order they are specified. The method of relaxation makes possible the application of a variety of schemes that alter the order.

Consider the system of equations

$$\begin{aligned}
x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 - z_1 &= 0 \\
b_{21}x_1 + x_2 + b_{23}x_3 + b_{24}x_4 - z_2 &= 0 \\
b_{31}x_1 + b_{32}x_2 + x_3 + b_{34}x_4 - z_3 &= 0 \\
b_{41}x_1 + b_{42}x_2 + b_{43}x_3 + x_4 - z_4 &= 0
\end{aligned}$$

where $b_{ij} = \dfrac{a_{ij}}{a_{ii}}$

$$z_i = \frac{y_i}{a_{ii}}$$

As in the Gauss and Gauss-Seidel iterative methods an initial set of values is selected for x_1, x_2, x_3, and x_4. Designating these initial values as $x_i^{(0)}$, the values $R_i^{(0)}$ obtained as a result of the initial substitution are:

$$\begin{aligned}
x_1^{(0)} + b_{12}x_2^{(0)} + b_{13}x_3^{(0)} + b_{14}x_4^{(0)} - z_1 &= R_1^{(0)} \\
b_{21}x_1^{(0)} + x_2^{(0)} + b_{23}x_3^{(0)} + b_{24}x_4^{(0)} - z_2 &= R_2^{(0)} \\
b_{31}x_1^{(0)} + b_{32}x_2^{(0)} + x_3^{(0)} + b_{34}x_4^{(0)} - z_3 &= R_3^{(0)} \\
b_{41}x_1^{(0)} + b_{42}x_2^{(0)} + b_{43}x_3^{(0)} + x_4^{(0)} - z_4 &= R_4^{(0)}
\end{aligned}$$

The relaxation procedure consists of estimating new values for the variables until all R_i's, called *residuals*, become negligible. The usual procedure is to select the largest residual R_i^k resulting from the kth iteration and calculate the change in x_i^k required to reduce R_i^k to zero. This change is

$$\Delta x_i^k = -R_i^k$$

and the new estimate for the variable is

$$x_i^{k+1} = x_i^k + \Delta x_i^k$$

Table 7.5 Solution by Gauss-Seidel iterative method

Iteration count	I_1	I_2	I_3
0	1.0	1.0	1.0
1	1.271132	1.205727	1.122111
2	1.205858	1.188588	1.127262
3	1.211052	1.184696	1.128145
4	1.212236	1.183912	1.128320
5	1.212474	1.183755	1.128355
6	1.212522	1.183724	1.128362

Table 7.6 Solution by relaxation method

Iteration count	I_1	I_2	I_3	R_1	R_2	R_3
0	1.0	1.0	1.0	−0.271132	−0.313844	−0.176581
1	1.0	1.313844	1.0	−0.174526	0	−0.099428
2	1.174526	1.313844	1.0	0	0.069594	−0.096920
3	1.174526	1.313844	1.096920	0.001546	0.103856	0
4	1.174526	1.209988	1.096920	−0.030423	0	−0.025531
5	1.204949	1.209988	1.096920	0	0.012131	−0.025094
6	1.204949	1.209988	1.122014	0.000400	0.021002	0
7	1.204949	1.188986	1.122014	−0.006064	0	−0.005163
8	1.211013	1.188986	1.122014	0	0.002419	−0.005076
9	1.211013	1.188986	1.127090	0.000081	0.004214	0
10	1.211013	1.184772	1.127090	−0.001216	0	−0.001036
11	1.212229	1.184772	1.127090	0	0.000484	−0.001019
12	1.212229	1.184772	1.128109	0.000016	0.000844	0
13	1.212229	1.183928	1.128109	−0.000244	0	−0.000207
14	1.212473	1.183928	1.128109	0	0.000098	−0.000203
15	1.212473	1.183928	1.128312	0.000003	0.000170	0
16	1.212473	1.183758	1.128312	−0.000050	0	−0.000042

Next, recompute the residuals,

$$\begin{aligned} R_1^{(1)} &= R_1^{(0)} + b_{12}\Delta I_2^{(0)} \\ &= -0.271132 + 0.307815(0.313844) \\ &= -0.271132 + 0.096606 = -0.174526 \\ R_3^{(1)} &= R_3^{(0)} + b_{32}\Delta I_2^{(0)} \\ &= -0.176581 + 0.245833(0.313844) \\ &= -0.176581 + 0.077153 = -0.099428 \end{aligned}$$

If the shunt elements e_s are included in forming the loop matrices, the number of elements of the network is increased by e_s. The total number of elements is, then, $e + e_s$ and the number of nodes is increased to $n + 1$. Consequently, the number of loops and the dimension of the loop matrices are increased by $e_s - 1$.

The different forms of network equations are summarized in Table 8.1.

Table 8.1 Network equations

Frame of reference	Parameter form: Impedance	Admittance
Bus	$\bar{E}_{BUS} = Z_{BUS}\bar{I}_{BUS}$	$\bar{I}_{BUS} = Y_{BUS}\bar{E}_{BUS}$
Loop	$\bar{E}_{LOOP} = Z_{LOOP}\bar{I}_{LOOP}$	$\bar{I}_{LOOP} = Y_{LOOP}\bar{E}_{LOOP}$

Bus loading equations

The real and reactive power at any bus p is

$$P_p - jQ_p = E_p^* I_p$$

and the current is

$$I_p = \frac{P_p - jQ_p}{E_p^*} \qquad (8.2.3)$$

where I_p is positive when flowing into the system.

In the formulation of the network equation, if the shunt elements to ground are included in the parameter matrix, then equation (8.2.3) is the total current at the bus. On the other hand, if the shunt elements are not included in the parameter matrix, the total current at bus p is

$$I_p = \frac{P_p - jQ_p}{E_p^*} - y_p E_p$$

where y_p is the total shunt admittance at the bus and $y_p E_p$ is the shunt current flowing from bus p to ground.

Line flow equations

After the iterative solution of bus voltages is completed, line flows can be calculated. The current at bus p in the line connecting bus p to q is

$$i_{pq} = (E_p - E_q)y_{pq} + E_p \frac{y'_{pq}}{2}$$

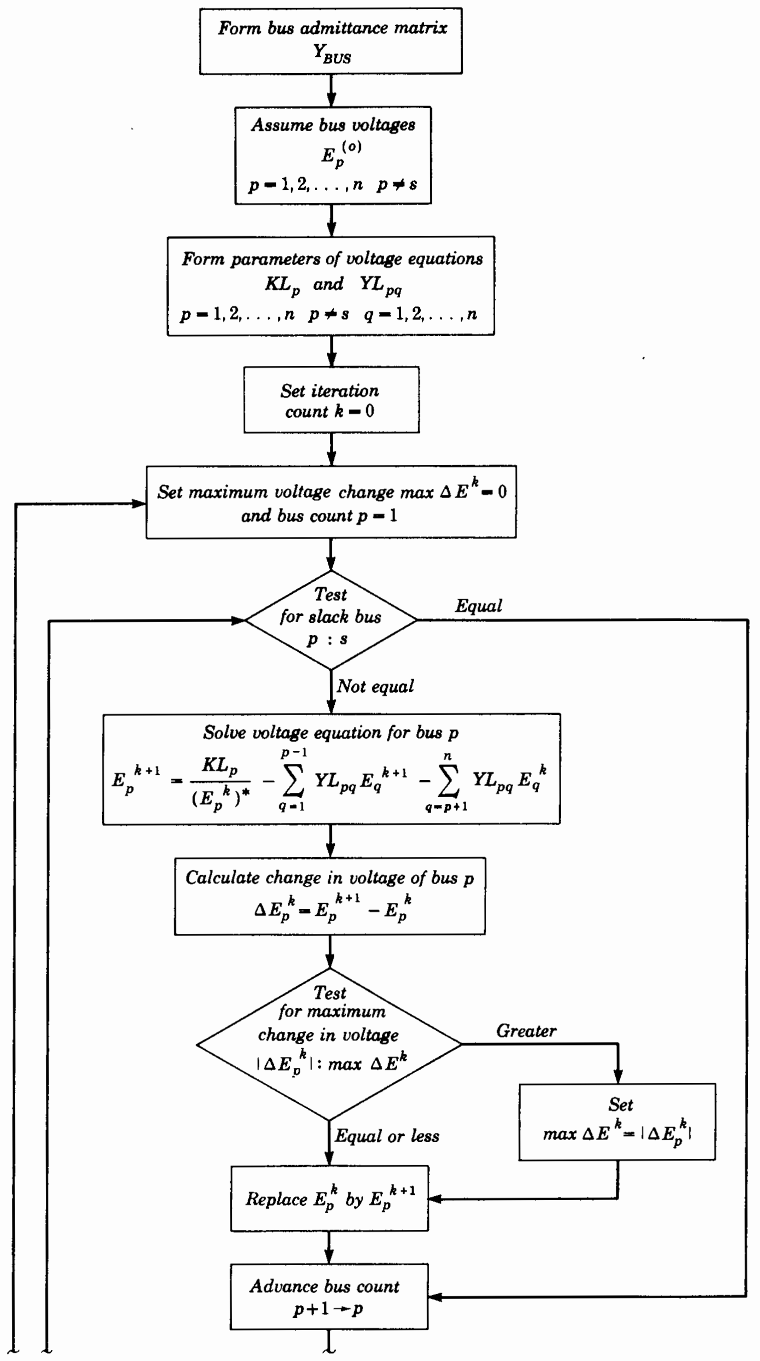

Fig. 8.3 Load flow solution by the Gauss-Seidel iterative method using Y_{BUS}.

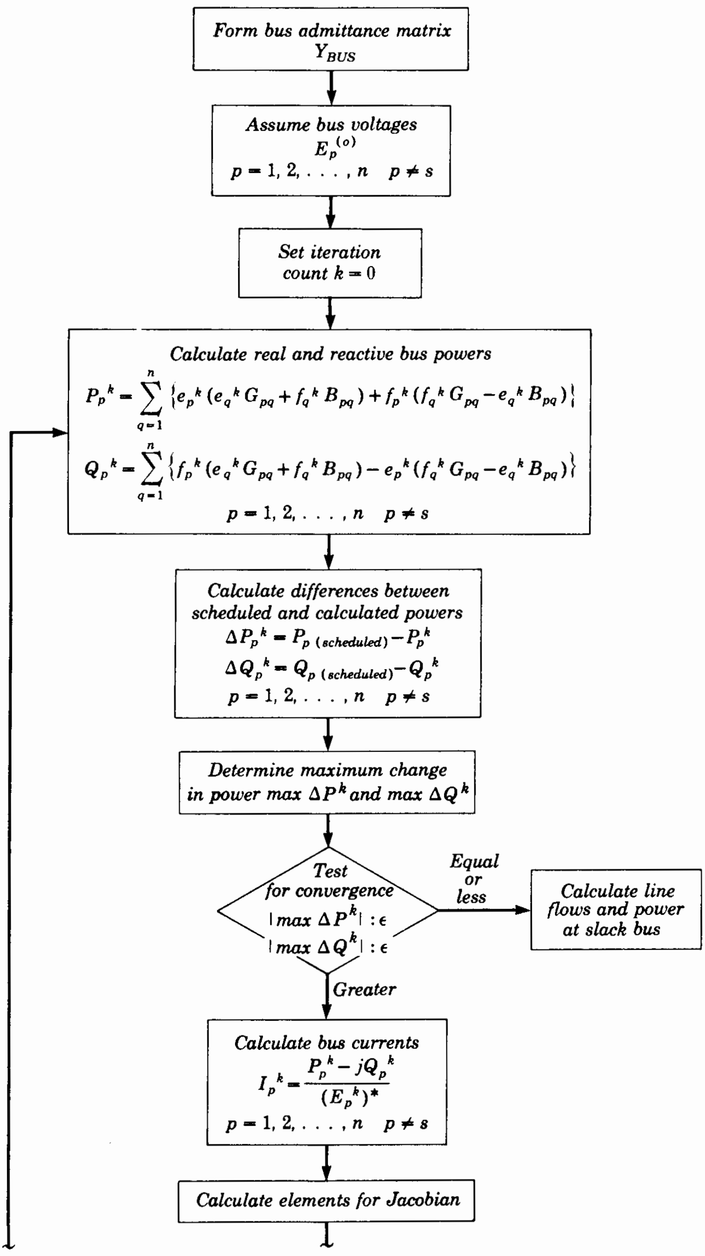

Fig. 8.5 **Load flow solution by the Newton-Raphson method using Y_{BUS}.**

For J_4:

$$\frac{\partial Q_p}{\partial |E_q|} = |E_p Y_{pq}| \sin(\theta_{pq} + \delta_p - \delta_q) \qquad q \neq p$$

$$\frac{\partial Q_p}{\partial |E_p|} = 2|E_p Y_{pp}| \sin \theta_{pp} + \sum_{\substack{q=1 \\ q \neq p}}^{n} |E_q Y_{pq}| \sin(\theta_{pq} + \delta_p - \delta_q)$$

Then the equation relating the changes in power to the changes in the voltage magnitudes and phase angles for the Newton-Raphson method is

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta|E| \end{bmatrix}$$

Approximations to Newton-Raphson method

In general, for a small change in the magnitude of bus voltage the real power at the bus does not change appreciably. Likewise, for a small change in the phase angle of the bus voltage the reactive power does not change appreciably. Therefore, using polar coordinates, a solution for the load flow problem can be obtained assuming the elements of the sub-matrices J_2 and J_3 are zero (Carpentier, 1963). The simplified matrix equation is

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_4 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta|E| \end{bmatrix}$$

Successful solutions can be obtained also by reevaluating the Jacobian in only the first few iterations.

When using rectangular coordinates, a solution to the load flow problem also can be obtained by neglecting the off-diagonal elements of the submatrices J_1, J_2, J_3, and J_4 of the Jacobian (Ward and Hale, 1956). This results in the following equations for the changes in real and reactive power at bus p:

$$\Delta P_p = \frac{\partial P_p}{\partial e_p} \Delta e_p + \frac{\partial P_p}{\partial f_p} \Delta f_p$$
$$= \Delta e_p(e_p G_{pp} - f_p B_{pp} + c_p) + \Delta f_p(e_p B_{pp} + f_p G_{pp} + d_p)$$

$$\Delta Q_p = \frac{\partial Q_p}{\partial e_p} \Delta e_p + \frac{\partial Q_p}{\partial f_p} \Delta f_p$$
$$= \Delta e_p(e_p B_{pp} + f_p G_{pp} - d_p) + \Delta f_p(-e_p G_{pp} + f_p B_{pp} + c_p)$$

$$p = 1, 2, \ldots, n-1$$

Table 8.3 ***Scheduled generation and loads and assumed bus voltages for sample system***

Bus code	*Assumed*	*Generation*		*Load*	
p	*bus voltage*	*Megawatts*	*Megavars*	*Megawatts*	*Megavars*
1	1.06 + *j*0.0	0	0	0	0
2	1.0 + *j*0.0	40	30	20	10
3	1.0 + *j*0.0	0	0	45	15
4	1.0 + *j*0.0	0	0	40	5
5	1.0 + *j*0.0	0	0	60	10

a. The equations for the Gauss-Seidel iterative solution using the bus code numbers given in Fig. 8.8, are

$$E_1 = 1.06 + j0.0$$

$$E_2^{k+1} = \frac{KL_2}{(E_2{}^k)^*} - YL_{21}E_1 - YL_{23}E_3{}^k - YL_{24}E_4{}^k - YL_{25}E_5{}^k$$

$$E_3^{k+1} = \frac{KL_3}{(E_3{}^k)^*} - YL_{31}E_1 - YL_{32}E_2^{k+1} - YL_{34}E_4{}^k$$

$$E_4^{k+1} = \frac{KL_4}{(E_4{}^k)^*} - YL_{42}E_2^{k+1} - YL_{43}E_3^{k+1} - YL_{45}E_5{}^k$$

$$E_5^{k+1} = \frac{KL_5}{(E_5{}^k)^*} - YL_{52}E_2^{k+1} - YL_{54}E_4^{k+1}$$

In order to calculate the parameters for these equations, it is necessary, first, to determine the elements of the bus admittance matrix from the transmission line and line charging admittances with ground as reference. The transmission line admittances, obtained by taking the reciprocal of the line impedances, are shown in Table 8.4 along with the total line charging admittance to ground at each bus. Since there is no mutual coupling in the representation of the system, the diagonal element of the bus admittance matrix for bus 1 is

$$Y_{11} = y_{12} + y_{13} + y_1$$

Prior to initiating the iterative process it is necessary to calculate the bus currents with the scheduled net bus powers and assumed initial bus voltages. From the values given in Table 8.3, the net bus powers in per unit are obtained and substituted in the equation

$$I_p^k = \frac{P_p - jQ_p}{(E_p^k)^*} - y_p E_p^k$$

The currents for all buses, except the reference, are

$$I_2^{(0)} = \frac{0.20 - j0.20}{1.0 - j0.0} - (0.0 + j0.085)(1.0 + j0.0) = 0.200 - j0.285$$

$$I_3^{(0)} = \frac{-0.45 + j0.15}{1.0 - j0.0} - (0.0 + j0.055)(1.0 + j0.0) = -0.450 + j0.095$$

$$I_4^{(0)} = \frac{-0.40 + j0.05}{1.0 - j0.0} - (0.0 + j0.055)(1.0 + j0.0) = -0.400 - j0.005$$

$$I_5^{(0)} = \frac{-0.60 + j0.10}{1.0 - j0.0} - (0.0 + j0.040)(1.0 + j0.0) = -0.600 + j0.060$$

The first step in the iterative process is to calculate a new estimate of the voltage for bus 2 by multiplying the first row of the bus impedance matrix by the vector of bus currents as follows:

$$\begin{aligned} E_2^{(1)} - E_1 &= Z_{22}I_2^{(0)} + Z_{23}I_3^{(0)} + Z_{24}I_4^{(0)} + Z_{25}I_5^{(0)} \\ &= (0.0168751 + j0.0505714)(0.200 - j0.285) \\ &\quad + (0.0125714 + j0.0377143)(-0.450 + j0.095) \\ &\quad + (0.0134286 + j0.0402857)(-0.400 - j0.005) \\ &\quad + (0.0157143 + j0.0471429)(-0.600 + j0.060) \\ &= -0.00888 - j0.05399 \end{aligned}$$

Since the voltage at the reference bus has been specified, as given in Table 8.3,

$$\begin{aligned} E_2^{(1)} &= (-0.00888 - j0.05399) + (1.060 + j0.0) \\ &= 1.05112 - j0.05399 \end{aligned}$$

The new current for bus 2 is

$$\begin{aligned} I_2^{(1)} &= \frac{P_2 - jQ_2}{(E_2^{(1)})^*} - y_2 E_2^{(1)} \\ &= \frac{0.20 - j0.20}{1.05112 + j0.05399} - (0.0 + j0.085)(1.05112 - j0.05399) \\ &= 0.17544 - j0.028887 \end{aligned}$$

The basic loop incidence matrix is

$C =$

e \ l	A	B	C
1	−1	−1	−1
2		−1	−1
3			−1
4	1	1	1
5	1		
6		1	
7			1

The loop admittance matrix is

$Y_{LOOP} =$

	A	B	C
A	1.07143 − j3.21429	−0.47619 + j1.42857	−0.23809 + j0.71429
B	−0.47619 + j1.42857	1.06349 − j3.19048	−0.30159 + j0.90476
C	−0.23809 + j0.71429	−0.30159 + j0.90476	0.68254 − j2.04762

which was obtained by first forming the loop impedance matrix by singular transformation and then taking its inverse. The loop admittance matrix can be derived also from the bus impedance matrix by using the algorithm described in Sec. 4.5.

The first step in the iterative process is to calculate the bus currents with the scheduled bus powers and assumed initial bus voltages. The currents in per unit for all buses except the slack are determined from the equation

$$I_p^k = \frac{P_p - jQ_p}{(E_p^k)^*} - y_p E_p^k$$

The new link currents from the equation

$$\bar{\imath}_l^k = \bar{\imath}_l^{k-1} + \bar{I}_{LOOP}^k$$

are

$\bar{\imath}_l^{(0)} =$

5	$0.24286 - j0.04786$
6	$0.27429 - j0.04029$
7	$0.53714 - j0.06014$

The new currents in all elements are, then,

$\bar{\imath}^{(0)} =$

1	$0.39572 - j0.00171$
2	$0.13857 + j0.04543$
3	$0.06286 + j0.00014$
4	$0.85428 + j0.13671$
5	$0.24286 - j0.04786$
6	$0.27429 - j0.04029$
7	$0.53714 - j0.06014$

These values replace the previous estimated flows.

and the diagonal elements are

$$\frac{\partial |E_p|^2}{\partial f_p} = 2f_p$$

The change in the square of the voltage magnitude at bus p is

$$\Delta|E_p{}^k|^2 = \{|E_p|_{(\text{scheduled})}\}^2 - |E_p{}^k|^2$$

If sufficient reactive capability is not available to hold the desired magnitude of bus voltage the reactive power must be fixed at a limit. In this case the bus is treated as a load bus with fixed reactive power.

In the Gauss-Seidel method using Z_{BUS} a correction to the reactive bus power can be calculated in order to provide the scheduled voltage (Brown, Carter, Happ, and Person, 1968). From the performance equation, the voltage at bus p is

$$e_p^{k+1} + jf_p^{k+1} = Z_{p1}I_1^{k+1} \cdots + Z_{pp}I_p{}^k \cdots + Z_{pn}I_n{}^k$$

The current at bus p can be corrected by ΔI_p to obtain

$$e_p + jf_p = Z_{p1}I_1^{k+1} \cdots + Z_{pp}(I_p{}^k + \Delta I_p{}^k) \cdots + Z_{pn}I_n{}^k$$

where e_p and f_p satisfy the equation (8.6.2). Subtracting the two voltage equations,

$$\Delta I_p{}^k = \frac{(e_p + jf_p) - (e_p^{k+1} + jf_p^{k+1})}{Z_{pp}}$$

or

$$\Delta I_p{}^k = \frac{(e_p^{k+1} + jf_p^{k+1})}{Z_{pp}}\left\{\frac{e_p + jf_p}{e_p^{k+1} + jf_p^{k+1}} - 1\right\}$$

Assuming that the phase angles of the scheduled voltage and E_p^{k+1} are equal, then,

$$\Delta I_p{}^k = \frac{(e_p^{k+1} + jf_p^{k+1})}{Z_{pp}}\left\{\frac{|E_p|_{(\text{scheduled})}}{|E_p^{k+1}|} - 1\right\}$$

The corresponding correction in the reactive power is

$$\Delta Q_p{}^k = -\text{Im}\{(E_p^{k+1})^*\Delta I_p{}^k\}$$

If the new reactive power

$$Q_p^{k+1} = Q_p{}^k + \Delta Q_p{}^k$$

is within the capability limits of the reactive source, then the new bus current is

$$I_p^{k+1} = \frac{P_p - jQ_p^{k+1}}{(E_p^{k+1})^*}$$

Substituting for A from equation (8.7.7) and solving for C,

$$C = y_{pq} - \frac{y_{pq}}{a}$$

$$= \left(1 - \frac{1}{a}\right) y_{pq}$$

Equating the current from equations (8.7.2) and (8.7.5) and substituting for A from (8.7.7),

$$(E_p - aE_q)\frac{y_{pq}}{a^2} = (E_p - E_q)\frac{y_{pq}}{a} + E_pB$$

Solving for B,

$$B = \frac{(E_p - aE_q)\dfrac{y_{pq}}{a^2} - (E_p - E_q)\dfrac{y_{pq}}{a}}{E_p}$$

$$= \frac{y_{pq}}{a^2} - \frac{y_{pq}}{a}$$

$$= \frac{1}{a}\left(\frac{1}{a} - 1\right) y_{pq}$$

The equivalent π circuit with its parameters expressed in terms of the turns ratio a and the transformer admittance are shown in Fig. 8.12c.

When the off-nominal turns ratio is represented at bus p for a transformer connecting p and q, the self-admittance at bus p is

$$Y_{pp} = y_{p1} \cdots + \frac{y_{pq}}{a} \cdots + y_{pn} + \frac{1}{a}\left(\frac{1}{a} - 1\right) y_{pq}$$

$$= y_{p1} + y_{p2} \cdots + \frac{y_{pq}}{a^2} \cdots + y_{pn}$$

The mutual admittance from p to q is

$$Y_{pq} = -\frac{y_{pq}}{a}$$

The self-admittance at bus q is

$$Y_{qq} = y_{q1} \cdots + \frac{y_{qp}}{a} \cdots + y_{qn} + \left(1 - \frac{1}{a}\right) y_{qp}$$

$$= y_{q1} \cdots + y_{qp} \cdots + y_{qn}$$

and is unchanged. The mutual admittance from q to p is

$$Y_{qp} = -\frac{y_{qp}}{a}$$

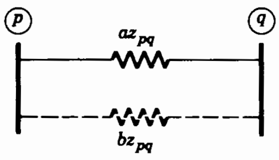

Fig. 8.13 Element added to network to reflect change in transformer tap setting

Solving for bz_{pq},

$$bz_{pq} = -\frac{a(a + \Delta a)}{\Delta a} z_{pq}$$

The change in tap setting of any transformer requires that every element of the Z_{BUS} matrix be recomputed. To avoid these extensive calculations an alternative equivalent can be used, in which the series impedance is made equal to the original transformer impedance and the shunt elements are varied to correspond to tap changes (Gupta and Humphrey Davies 1961).

Letting $A = y_{pq}$ and equating the corresponding terminal currents from equations (8.7.2) and (8.7.5) for the transformer and its equivalent, respectively, then

$$(E_p - E_q)y_{pq} + E_pB = (E_p - aE_q)\frac{y_{pq}}{a^2}$$

Solving for B,

$$B = \left\{(E_p - aE_q)\frac{y_{pq}}{a^2} - (E_p - E_q)y_{pq}\right\}\frac{1}{E_p}$$
$$= \left\{\left(\frac{1}{a^2} - 1\right) - \left(\frac{1}{a} - 1\right)\frac{E_q}{E_p}\right\}y_{pq}$$
$$= \left(\frac{1}{a} - 1\right)\left\{\left(\frac{1}{a} + 1\right) - \frac{E_q}{E_p}\right\}y_{pq} \qquad (8.7.8)$$

Similarly, equating the terminal currents I_q from equations (8.7.4) and (8.7.6) with $A = y_{pq}$,

$$(E_q - E_p)y_{pq} + E_qC = (aE_q - E_p)\frac{y_{pq}}{a}$$

Solving for C,

$$C = \left\{(aE_q - E_p)\frac{y_{pq}}{a} - (E_q - E_p)y_{pq}\right\}\frac{1}{E_q}$$
$$= \left(1 - \frac{1}{a}\right)\frac{y_{pq}E_p}{E_q} \qquad (8.7.9)$$

The shunt admittances, (8.7.8) and (8.7.9), at buses p and q, respectively, are a function of the voltages E_p and E_q. The bus loading equations are, then,

$$I_p = \frac{P_p - jQ_p}{E_p^*} - y_p E_p - \left(\frac{1}{a} - 1\right)\left\{\left(\frac{1}{a} + 1\right) - \frac{E_q}{E_p}\right\} y_{pq} E_p$$

$$I_q = \frac{P_q - jQ_q}{E_q^*} - y_q E_q - \left(1 - \frac{1}{a}\right) y_{pq} E_p$$

Phase shifting transformers

A phase shifting transformer can be represented in load flow studies by its impedance, or admittance, connected in series with an ideal autotransformer having a complex turns ratio, as shown in Fig. 8.14. Then the terminal voltages E_p and E_s are related by

$$\frac{E_p}{E_s} = a_s + jb_s \tag{8.7.10}$$

Since there is no power loss in an ideal autotransformer,

$$E_p^* i_{pr} = E_s^* i_{sq} \tag{8.7.11}$$

It follows from equations (8.7.10) and (8.7.11) that

$$\frac{i_{pr}}{i_{sq}} = \frac{E_s^*}{E_p^*}$$

$$= \frac{1}{a_s - jb_s}$$

Since

$$i_{sq} = (E_s - E_q) y_{pq}$$

then

$$i_{pr} = (E_s - E_q) \frac{y_{pq}}{a_s - jb_s}$$

Substituting for E_s from equation (8.7.10),

$$i_{pr} = \{E_p - (a_s + jb_s) E_q\} \frac{y_{pq}}{a_s^2 + b_s^2} \tag{8.7.12}$$

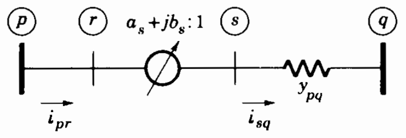

***Fig. 8.14** Phase shifting transformer representation.*

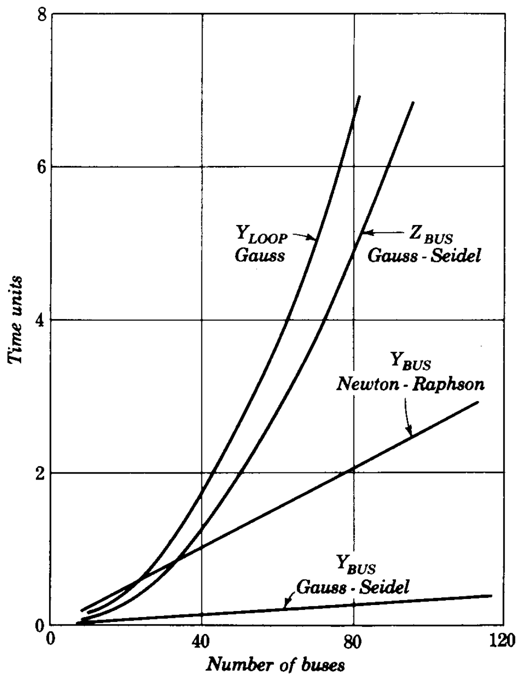

Fig. 8.16 Time per iteration for load flow methods.

The rate of convergence of the Gauss-Seidel method using the bus admittance matrix is slow, requiring a relatively greater number of iterations to obtain a solution than the Newton-Raphson method and the methods using the bus impedance or loop admittance matrices. In addition, the number of iterations for the Gauss-Seidel method increased directly as the number of buses of the network, whereas the number of iterations for the other methods remained relatively constant, independent of system size. A significant increase in the rate of convergence can be obtained for the Gauss-Seidel method using the bus admittance matrix by applying acceleration factors.

The optimum values of acceleration factors for a load flow solution are difficult to calculate; however, they can be determined empirically. The selection of values for α and β, the acceleration factors for the real and imaginary components of voltage, depends on the characteristics of the network and the method of solution. The effectiveness of different acceleration factors on the rate of convergence for the principal methods presented is shown in Fig. 8.17. A system of 30 buses and 41 lines was used for this analysis.

The tolerance required to obtain a solution varies with the different and removal of lines and transformers require that the network matrix be modified.

When the bus admittance matrix is used, it is necessary to recompute only those elements of the matrix that are associated with the terminals of the lines or transformers that are being changed. Because relatively few matrix elements are associated with any one bus, network changes can be effected simply and quickly. However, in order to modify the bus impedance matrix it is necessary to use the algorithm. Network changes that result in adding a new link make it necessary that all elements of the network matrix be modified. An algorithm would have to be developed in order to provide a means of modifying the loop admittance matrix.

The selection of initial values for bus voltages can have a marked effect on solution time. When a series of load flow calculations are performed, the usual procedure is to use the final calculated bus voltages of each case as the initial voltages for the next case. This tends to reduce the number of iterations, particularly when there are only minor changes in system conditions.

The actual computer time required for a load flow solution is dependent also on the speed of the digital computer and the efficiency of the program. The time units used in the comparison, therefore, would differ considerably from one digital computer to another. In general, however, each time unit is equal to about 1 sec for a medium size computer and to 0.1 sec or less for a large-scale computer.

8.10 Description of load flow program

Large-scale load flow computer programs incorporate many automatic features to facilitate their use in power system planning, operating, and interconnection studies. The principal objectives of these features are to make maximum use of the computer's capability and to minimize the number of manual operations required by the engineer in specifying and maintaining system data for the initial and subsequent load flow cases.

The American Electric Power Load Flow Program consists of an integrated set of computer programs to perform load flow calculations and associated data processing. The principal components of this program are:

Input

The input program provides the ability to read into the computer the power system data for a load flow calculation. This data is converted to the proper computer representation and stored in memory in the specified locations.

tolerances of 0.01 per unit for the changes in real and reactive bus powers
c. Gauss-Seidel using Z_{BUS} with voltage tolerances of 0.01 and 0.01 per unit

8.2 With tolerances of 0.01 per unit for the changes in real and reactive bus powers, obtain a load flow solution for the sample system shown in Fig. 8.23 using the following methods:
a. Newton-Raphson using Y_{BUS} in rectangular coordinates. Assume the off-diagonal elements of the submatrices J_1, J_2, J_3, and J_4 of the Jacobian to be zero.
b. Newton-Raphson using Y_{BUS} in polar coordinates.
c. Newton-Raphson using Y_{BUS} in polar coordinates. Assume the submatrices J_2 and J_3 of the Jacobian to be zero.
Compare the convergence characteristics of these techniques and that of the method used in Prob. 8.1, part *b*.

8.3 Add to the sample system shown in Fig. 8.23 a second circuit from bus 1 to bus 3 with an impedance of $0.02 + j0.06$. Assume a fixed reactive generation of 25 megavars at bus 2 instead of maintaining voltage at that bus. Using the data given in Tables 8.19 and 8.20, obtain a load flow solution by the Gauss method using Y_{LOOP}. Let the loop voltage tolerances be 0.01 and 0.01.

Table 8.19 Impedances for sample system for Prob. 8.1

Bus code p–q	Impedance z_{pq}	Line charging $y'_{pq}/2$
1–2	$0.08 + j0.24$	0
1–3	$0.02 + j0.06$	0
2–3	$0.06 + j0.18$	0

Table 8.20 Scheduled generation and loads and assumed bus voltages for sample system for Prob. 8.1

Bus code p	Assumed bus voltage	Generation		Load	
		Megawatts	Megavars	Megawatts	Megavars
1	$1.05 + j0$	0	0	0	0
2	$1.0 + j0$	20	0	50	20
3	$1.0 + j0$	0	0	60	25

8.4 The sample power system shown in Fig. 8.24 is composed of a tap changing under load transformer with an impedance of $0 + j0.03$. The load at bus 2 is 200 megawatts and 100 megavars. Let bus 1 be the slack and its voltage be $1.0 + j0$ per unit. Assume an initial tap setting of 1.0 and determine the required tap setting to hold a voltage magnitude of 1.0 per unit, within ± 0.005 per unit, at bus 2. Use the Gauss-Seidel method with Y_{BUS} and a $\frac{5}{8}$ percent step per iteration for the tap change.

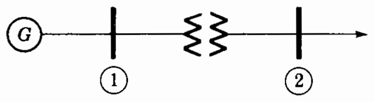

Fig. 8.24 Sample power system for Prob. 8.4.

8.5 The load flow data for the sample power system shown in Fig. 8.25 is given in Tables 8.21 and 8.22. The voltage magnitude at bus 2 is to be held at 1.0 per unit by means of the synchronous condenser at bus 3. The maximum and minimum reactive power limits of the condenser are 50 and -10 megavars, respectively. With bus 1 as the slack, use the Gauss-Seidel method and the bus admittance matrix to obtain a load flow solution. Use voltage tolerances of 0.001 and 0.001 per unit and acceleration factors of 1.4 and 1.4.

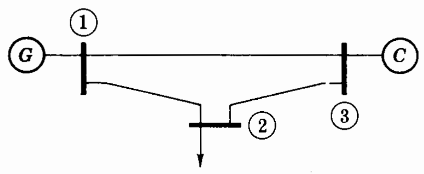

Fig. 8.25 Sample power system for Prob. 8.5.

Table 8.21 Impedances for sample system for Prob. 8.5

Bus code p–q	*Impedance* z_{pq}	*Line charging* $y'_{pq}/2$
1–2	$0 + j0.05$	0
1–3	$0 + j0.10$	0
2–3	$0 + j0.05$	0

Table 8.22 Scheduled generation and loads and assumed bus voltages for sample system for Prob. 8.5

Bus code p	Assumed bus voltage	Generation		Load	
		Megawatts	Megavars	Megawatts	Megavars
1	$1.03 + j0$	0	0	0	0
2	$1.00 + j0$	0	0	200	100
3	$1.00 + j0$	0	0	0	0

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Table 9.4 Solution by Milne's method

n	Time t_n	Voltage e_n	Current (predicted) i_n	i'_n	Current (corrected) i_n
4	0.100	0.500	0.02418	0.47578	0.02419
5	0.125	0.625	0.03748	0.58736	0.03748
6	0.150	0.750	0.05353	0.69601	0.05353
7	0.175	0.875	0.07226	0.80161	0.07226
8	0.200	1.000	0.09359	0.90395	0.09358
9	0.225	1.000	0.11742	0.87772	0.11639
				0.87888	0.11640†
10	0.250	1.000	0.13543	0.85712	0.13755
				0.85464	0.13753†
11	0.275	1.000	0.16021	0.82745	0.15911
				0.82881	0.15912†
12	0.300	1.000	0.17894	0.80387	0.17898
				0.80382	0.17898†

† Second corrected value obtained by iteration.

by substituting this corrected value in the differential equation to obtain $i'_9 = 0.87888$. This in turn was used in the corrector formula to obtain the second estimate for $i_9 = 0.11640$, which checks the previous corrected value. An iteration was performed in all subsequent steps to assure the desired accuracy.

e. The equation used for Picard's method to generate an approximating function for i, near $i_0 = 0$, is

$$i = i_0 + \int_0^t (e(t) - i - 3i^3)\,dt$$

Substituting $e(t) = 5t$ and the initial value $i_0 = 0$,

$$i^{(1)} = \int_0^t 5t\,dt = \frac{5t^2}{2}$$

Then, substituting $i^{(1)}$ for i in the integral equation,

$$i^{(2)} = \int_0^t \left(5t - \frac{5t^2}{2} - \frac{375t^6}{8}\right) dt$$
$$= \frac{5t^2}{2} - \frac{5t^3}{6} - \frac{375t^7}{56}$$

method of the first type. The methods of Euler, Runge-Kutta, and Milne are examples of the second type.

The principal difficulties that arise from methods approximating y by a function, such as Picard's method, occur in the repeated explicit integrations that must be performed to obtain a satisfactory function. Hence these methods are impractical in most cases and are seldom used.

The methods of the second type require simple arithmetic operations and thus are applicable for a computer solution of differential equations. In general, the simpler relations require the use of smaller intervals for the independent variable whereas the more complex methods can employ relatively larger intervals without sacrificing the accuracy of the solution. Euler's method is the simplest, but unless a very small interval is used it is too inaccurate to be practical. The modified method of Euler is also simple to apply and has the additional advantage that systematic checking is inherent in the process of obtaining improved estimates for y. This method is of limited accuracy, however, and requires the use of small intervals for the independent variable. The Runge-Kutta method requires a larger number of arithmetic operations, but the results are more accurate.

Milne's predictor-corrector method is less laborious than is the Runge-Kutta method and has comparable accuracy of order h^5. However, Milne's method requires four starting values for the dependent variable that must be obtained by some other method, such as the modified Euler or Runge-Kutta method, that is self-starting. For a computer application this requires programming a numerical method for starting the solution as well as Milne's method for continuing the solution. The use of different formulas for predicting and then correcting a value of y provides a systematic process for checking as well as correcting the initial estimate. If the difference between the predicted and corrected values is significant, the interval can be reduced. This capability in the Milne method is not available in the Runge-Kutta method.

Problems

9.1 Solve the differential equation

$$\frac{dy}{dx} = x^2 - y$$

for $0 \leq x \leq 0.3$, with the interval equal to 0.05 and initial values $x_0 = 0$ and $y_0 = 1$, by the following numerical methods:

a. Euler's
b. The modified Euler
c. Picard's

chapter 10
Transient stability studies

10.1 Introduction

Transient stability studies provide information related to the capability of a power system to remain in synchronism during major disturbances resulting from either the loss of generating or transmission facilities, sudden or sustained load changes, or momentary faults. Specifically, these studies provide the changes in the voltages, currents, powers, speeds, and torques of the machines of the power system, as well as the changes in system voltages and power flows, during and immediately following a disturbance. The degree of stability of a power system is an important factor in the planning of new facilities. In order to provide the reliability required by the dependence on continuous electric service, it is necessary that power systems be designed to be stable under any conceivable disturbance.

The ac network analyzer was used for transient stability studies to obtain the operating performance of the power network during a disturbance. The step-by-step calculations describing the operation of the machines were performed manually. The use of the digital computer to perform all computation for both the network and the machines was a natural extension of the digital load flow studies that proved so successful.

The performance of the power system during the transient period can be obtained from the network performance equations. The performance equation using the bus frame of reference in either the impedance or admittance form has been used in transient stability calculations.

In transient stability studies a load flow calculation is made first to obtain system conditions prior to the disturbance. In this calculation the network is composed of system buses, transmission lines, and trans-

10.4 Power system equations

Representation of loads

Power system loads, other than motors represented by equivalent circuits, can be treated in several ways during the transient period. The commonly used representations are either static impedance or admittance to ground, constant current at fixed power factor, constant real and reactive power, or a combination of these representations.

The constant power load is either equal to the scheduled real and reactive bus load or is a percentage of the specified values in the case of a combined representation. The parameters associated with static impedance and constant current representations are obtained from the scheduled bus loads and the bus voltages calculated from a load flow solution for the power system prior to a disturbance. The initial value of the current for a constant current representation is obtained from

$$I_{p0} = \frac{P_{Lp} - jQ_{Lp}}{E_p^*}$$

where P_{Lp} and Q_{Lp} are the scheduled bus loads and E_p is the calculated bus voltage. The current I_{p0} flows from bus p to ground, that is, to bus 0. The magnitude and power factor angle of I_{p0} remain constant.

The static admittance y_{p0}, used to represent the load at bus p, can be obtained from

$$(E_p - E_0)y_{p0} = I_{p0}$$

where E_p is the calculated bus voltage and E_0 is the ground voltage, equal to zero. Therefore

$$y_{p0} = \frac{I_{p0}}{E_p} \tag{10.4.1}$$

Multiplying both the dividend and divisor of equation (10.4.1) by E_p^* and separating the real and imaginary components,

$$g_{p0} = \frac{P_{Lp}}{e_p^2 + f_p^2} \quad \text{and} \quad b_{p0} = \frac{Q_{Lp}}{e_p^2 + f_p^2}$$

where

$$y_{p0} = g_{p0} - jb_{p0}$$

Network performance equations

The network performance equations used for load flow calculations can be applied to describe the performance of the network during a transient

Subsequent voltages for these buses are calculated from the differential equations describing the performance of the machines.

During the iterative calculation the magnitudes and phase angles of the bus voltages behind the machine equivalent admittances are held constant. If a three-phase fault is simulated, the voltage of the faulted bus is set to zero and held constant.

If the bus impedance matrix is used for a transient stability study, ground is usually taken as reference because all network bus voltages, except at the faulted bus, change during the transient period (Brown, Happ, Person, and Young, 1965). To eliminate the need to modify the bus impedance matrix for a change in the reference bus, ground is used also as reference in the prefault load flow calculation.

When ground is used as reference for the load flow calculation and the loads are represented solely as current sources, the bus impedance matrix will include only the capacitor, reactor, and line charging elements to ground. In this case the bus impedance matrix is ill-conditioned and convergence of the solution usually is not obtained. On the other hand, if the loads are represented solely as impedances to ground to improve the convergence characteristic, then these impedances and the bus impedance matrix must be modified during the iterative solution for changes in bus voltages. To overcome this difficulty only a portion of each bus load is represented as an impedance to ground. The remaining portion of the load can be represented as a current source which varies with the bus voltage so that the total bus current satisfies the scheduled load power.

After the load flow solution is obtained, the bus impedance matrix must be modified to include the new network elements representing the machines and to account for changes in the representation of loads. These modifications can be made by using the algorithm described in Secs. 4.2 and 4.3. Each element representing a machine is a branch to a new bus, and each element representing a load change is a link to ground.

The iteration formula for the performance of the network during the transient period using ground as reference is

$$E_p^{k+1} = \sum_{q=1}^{n+m} Z_{pq} I_q \qquad \begin{array}{l} p = 1, 2, \ldots, n \\ p \neq f \end{array}$$

where n is the number of network buses, m is the number of buses behind the equivalent machine impedances, and bus f is the faulted bus. The current vector I_q is composed of load currents from either the constant current or constant power representation and the currents obtained from machine equivalent circuits.

In the application of the bus impedance matrix, only those rows and simplified machine representation, are determined from

$$\Delta\delta_{i(t+\Delta t)} = \tfrac{1}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i})$$
$$\Delta\omega_{i(t+\Delta t)} = \tfrac{1}{6}(l_{1i} + 2l_{2i} + 2l_{3i} + l_{4i}) \qquad i = 1, 2, \ldots, m$$

The k's and l's are the changes in δ_i and ω_i, respectively, obtained using derivatives evaluated at predetermined points. Then,

$$\delta_{i(t+\Delta t)} = \delta_{i(t)} + \tfrac{1}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i})$$
$$\omega_{i(t+\Delta t)} = \omega_{i(t)} + \tfrac{1}{6}(l_{1i} + 2l_{2i} + 2l_{3i} + l_{4i}) \qquad (10.5.3)$$

The initial estimates of changes are obtained from

$$k_{1i} = (\omega_{i(t)} - 2\pi f)\,\Delta t$$
$$l_{1i} = \frac{\pi f}{H_i}(P_{mi} - P_{ei(t)})\,\Delta t \qquad i = 1, 2, \ldots, m$$

where $\omega_{i(t)}$ and $P_{ei(t)}$ are the machine speeds and air-gap powers at time t. The second set of estimates of changes in δ_i and ω_i are obtained from

$$k_{2i} = \left\{\left(\omega_{i(t)} + \frac{l_{1i}}{2}\right) - 2\pi f\right\}\Delta t$$
$$l_{2i} = \frac{\pi f}{H_i}(P_{mi} - P_{ei}^{(1)})\,\Delta t \qquad i = 1, 2, \ldots, m$$

where $P_{ei}^{(1)}$ are the machine powers when the internal voltage angles are $\delta_{i(t)} + (k_{1i}/2)$. Thus, before l_{2i} can be calculated, new components for the voltages for the internal machine buses must be calculated from

$$e_i'^{(1)} = |E_i'| \cos\left(\delta_{i(t)} + \frac{k_{1i}}{2}\right)$$
$$f_i'^{(1)} = |E_i'| \sin\left(\delta_{i(t)} + \frac{k_{1i}}{2}\right) \qquad i = 1, 2, \ldots, m$$

Then, the network equations are solved to obtain bus voltages for the calculation of machine powers $P_{ei}^{(1)}$.

The third set of estimates are obtained from

$$k_{3i} = \left\{\left(\omega_{i(t)} + \frac{l_{2i}}{2}\right) - 2\pi f\right\}\Delta t$$
$$l_{3i} = \frac{\pi f}{H_i}(P_{mi} - P_{ei}^{(2)})\,\Delta t \qquad i = 1, 2, \ldots, m$$

where $P_{ei}^{(2)}$ are obtained from a second solution of the network equations with the internal voltage angles equal to $\delta_{i(t)} + (k_{2i}/2)$ and the compo-

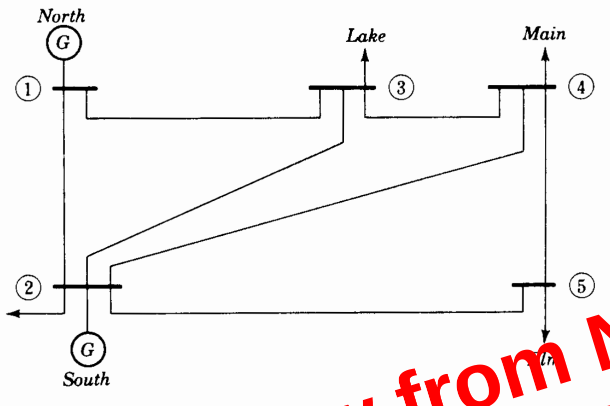

***Fig. 10.8** Sample system for transient stability calculations.*

Problem

Using the bus admittance matrix and the Gauss-Seidel iterative method for the solution of the network equations and the modified Euler method for the solution of the swing equations:

a. Determine the effects on the sample power system shown in Fig. 10.8 of a three-phase fault on bus 2 for a duration of 0.1 sec.
b. Determine the effects of the fault on bus 2 for a duration of 0.2 sec.

Solution

The results of the load flow calculation prior to the fault are given in Table 10.1. The inertia constants, direct-axis transient reactances, and equivalent admittances of the generators at buses 1 and 2 in per unit on a 100,000 kva base are given in Table 10.2.

***Table 10.1** Bus voltages, generation, and loads from load flow calculation prior to fault*

Bus code p	Bus voltages E_p	Generation		Load	
		Megawatts	Megavars	Megawatts	Megavars
1	$1.06000 + j0.00000$	129.565	−7.480	0.0	0.0
2	$1.04621 - j0.05128$	40.0	30.0	20.0	10.0
3	$1.02032 - j0.08920$	0.0	0.0	45.0	15.0
4	$1.01917 - j0.09506$	0.0	0.0	40.0	5.0
5	$1.01209 - j0.10906$	0.0	0.0	60.0	10.0

where the bus voltage is obtained from the load flow solution and is given in Table 10.1. The line parameter YL_{21} is

$$YL_{21} = \frac{-5.00000 + j15.00000}{11.01562 - j33.17281}$$
$$= -0.45235 - j0.00052$$

The YL_{pq}'s for all elements are given in Table 10.3.

The voltages behind the equivalent admittances representing the machines are obtained from the equation

$$E'_i = E_{ti} + jx'_{di}I_{ti} \qquad i = n + 1, n + 2, \ldots, n + m$$

where

$$I_{ti} = \frac{P_{ti} - jQ_{ti}}{E^*_{ti}}$$

and n is the number of buses of the network and m is the number of machines. For the machine at bus 1

$$E_6 = 1.06 + j0.0 + j0.25\left(\frac{1.29565 + j0.07480}{1.06 - j0.0}\right)$$
$$= 1.04236 + j0.30558$$

Table 10.3 Line parameters for transient stability representation of sample system

Bus code p–q	YL_{pq}
1–2	$-0.67074 - j0.03560$
1–3	$-0.16769 - j0.00890$
1–6	$-0.16383 + j0.04512$
2–1	$-0.45235 - j0.00052$
2–3	$-0.15078 - j0.00017$
2–4	$-0.15078 - j0.00017$
2–5	$-0.22618 - j0.00026$
2–7	$-0.01810 + j0.00601$
3–1	$-0.09625 + j0.00089$
3–2	$-0.12833 + j0.00119$
3–4	$-0.77000 + j0.00711$
4–2	$-0.12866 + j0.00115$
4–3	$-0.77198 + j0.00687$
4–5	$-0.09650 + j0.00086$
5–2	$-0.65236 + j0.02866$
5–4	$-0.32618 + j0.01433$

Similarly, for the machine at bus 2,

$$\delta_{7(0.02)}^{(1)} = 0.32097 + \left(\frac{0.0 + 1.50797}{2}\right) 0.02$$
$$= 0.32097 + 0.01508$$
$$= 0.33605$$

The internal voltage angles in degrees at $t + \Delta t = 0.02$ are

$$\delta_{6(0.02)}^{(1)} = 0.28598 \left(\frac{180}{\pi}\right) = 16.38540°$$

and

$$\delta_{7(0.02)}^{(1)} = 0.33605 \left(\frac{180}{\pi}\right) = 19.25420°$$

At $t + \Delta t = 0.02$ the final components of voltages behind the machine equivalent admittances are

$$e_6'^{(1)} = 1.08623 \cos (16.38540)$$
$$= 1.04212$$
$$f_6'^{(1)} = 1.08623 \sin (16.38540)$$
$$= 0.30641$$

and

$$e_7'^{(1)} = 1.58426 \cos (19.25420)$$
$$= 1.49564$$
$$f_7'^{(1)} = 1.58426 \sin (19.25420)$$
$$= 0.52243$$

Then the network equations are solved to obtain the final system voltages at $t + \Delta t = 0.02$. The voltages obtained from this calculation are given in Table 10.5.

***Table 10.5** Bus voltages of sample system at* $t + \Delta t = 0.02$

Bus code p	*Bus voltage* E_p
1	0.19258 + j0.00353
2	0.0 + j0.0
3	0.04815 − j0.00114
4	0.03845 − j0.00133
5	0.01249 − j0.00097

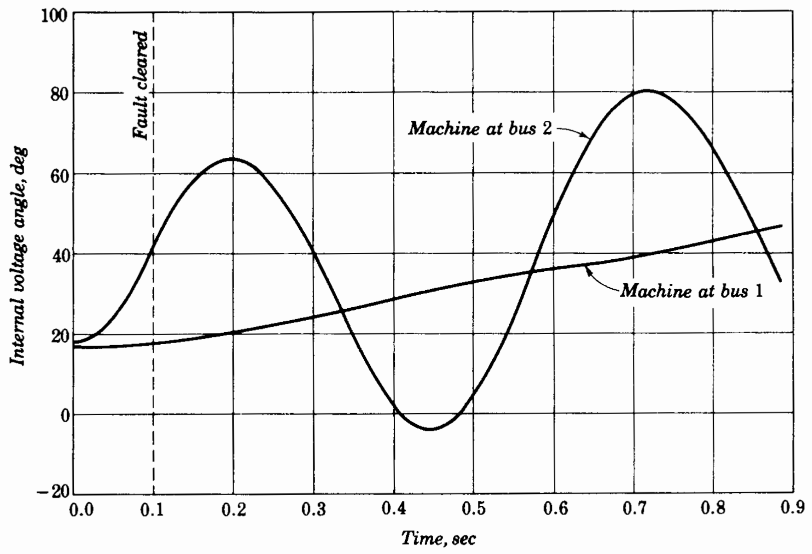

Fig. 10.10 ***Internal voltage angle of machine with respect to time for a fault duration of 0.1 sec.***

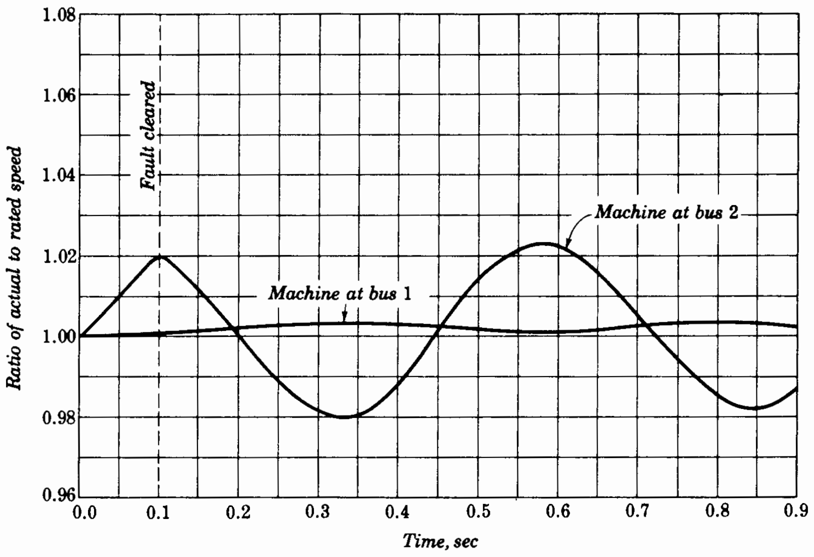

Fig. 10.11 ***Ratio of actual to rated speed of machine with respect to time for a fault duration of 0.1 sec.***

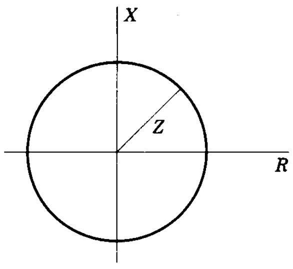

Fig. 10.17 Operating characteristic of distance relay plotted on an RX diagram.

circuit breaker operations to disconnect the faulted equipment. The design of a protective relaying system must assure proper operation so as not to disconnect additional equipment that would aggravate the effects of the disturbance and it must assure that the faulted equipment is cleared sufficiently fast to mitigate the effects of the fault. In addition, the relaying system must not limit the design capability of the generation and transmission facilities.

An important type of relay that is used for high-voltage transmission line protection is the distance relay. This relay responds to the ratio of measured voltage to measured current which can be expressed as an impedance. A convenient means of showing the operating characteristic of a distance relay is with an *RX* diagram on which a circle is drawn with the radius equal to the impedance setting as shown in Fig. 10.17. When the value of the impedance seen by the relay falls within the circle, the relay will operate.

To provide adequate primary and backup protection, distance relays have three units. The operating characteristic of each unit can be adjusted independently. Furthermore, the proper functioning of distance relays requires the capability to distinguish direction. This is provided by either a directional unit, as in the impedance-type distance relay, or is inherent in the operating characteristics, as in the mho-type distance relay. The operating characteristics of these two relays are shown in Fig. 10.18. The circles associated with the three units are labeled zone 1, zone 2, and zone 3.

When a fault occurs and the value of impedance seen by the relay falls within zone 1 and above the characteristic of the directional unit of the impedance type, the zone 1 contacts will close and trip the circuit breaker immediately. In this case all three units will operate because zone 1 is the smallest circle. When the impedance falls only within zones 2 and 3, or zone 3, the contacts of the associated units will close and energize a timer. At a specified time setting, the timer will close a second set of contacts associated with zone 2. If the first set of contacts

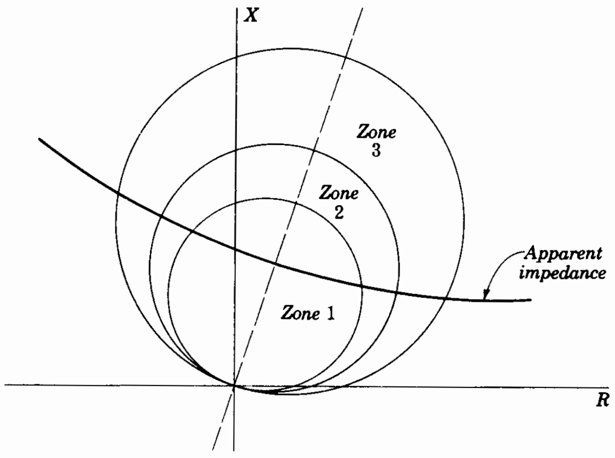

Fig. 10.19 Trajectory of apparent impedance during a power swing.

to operate. The operation of the relay system can be tested for various system disturbances by calculating during the step-by-step transient calculations the apparent impedance, that is, the impedance seen by the relay. The apparent impedance calculated at each time increment can be compared to the operating characteristics of the relay. A convenient means of making this comparison is to plot the impedance values on the RX diagram of the relay as shown in Fig. 10.19.

The apparent impedance is calculated from the final results obtained from the network solution at time $t + \Delta t$. First the current in a specified transmission line p–q is calculated from

$$I_{pq} = (E_p - E_q)y_{pq}$$

Then, the apparent impedance for bus p is

$$Z_p = \frac{E_p}{I_{pq}}$$

or in complex form,

$$R_p + jX_p = \frac{e_p + jf_p}{a_{pq} + jb_{pq}}$$

where $$R_p = \frac{e_p a_{pq} + f_p b_{pq}}{a_{pq}^2 + b_{pq}^2}$$

$$X_p = \frac{f_p a_{pq} - e_p b_{pq}}{a_{pq}^2 + b_{pq}^2}$$