We just finished doing a substitution of the summation index. It is equivalent to replacing i by $l = i + 2$. This relation also implies that $i = 1 \;\Rightarrow\; l = 3$ and $i = k \;\Rightarrow\; l = k + 2$. This is actually how you can determine the starting and the ending values of the new index.

Now we can solve this sum using Rule 4 and the known formula.

$$S = \sum_{i=1}^{k}(i+2)^3 = \sum_{l=3}^{k+2} l^3 = \sum_{l=1}^{k+2} l^3 - \sum_{l=1}^{2} l^3 = \left[\frac{1}{2}(k+2)(k+3)\right]^2 - 1^2 - 2^3 = \left[\frac{1}{2}(k+2)(k+3)\right]^2 - 9.$$

2.6 Applications of sigma sum

The area under a curve

We know that the area of a rectangle with length l and width w is $A_{rect} = w \cdot l$.

Starting from this formula we can calculate the area of a triangle and a trapezoid. This is because a triangle and a trapezoid can be transformed into a rectangle (see Figure). Thus, for a triangle of height h and base length b

$$A_{trig} = \frac{1}{2}hb.$$

Similarly, for a trapezoid with base length b, top length t, and height h

$$A_{trap} = \frac{1}{2}h(t + b).$$

Following a very similar idea, the sum of a trapezoid-shaped pile of logs with t logs on top layer, b logs on the bottom layer, and a height of $h = b - t + 1$ layers (see figure) is

$$\sum_{i=t}^{b} i = t + (t+1) + \cdots + (b-1) + b = \frac{1}{2}h(t+b) = \frac{1}{2}(b - t + 1)(t + b). \tag{8}$$

Now returning to the problem of calculating the area. Another important formula is for the area of a circle of radius r.

$$A_{circ} = \pi r^2.$$

Now, once we learned sigma and/or integration, we can calculate the area under the curve of any function that is integrable.

Example 10: Calculate the area under the curve $y = x^2$ between 0 and 2 (see figure).

Solution 10: Remember always try to reduce a problem that you do not know how to solve into a problem that you know how to solve.

respectively, the lower (or the starting) limit and the upper (or the ending) limit of the integral. Thus,

$$\int_a^b f(x)dx$$

means *integrate the function $f(x)$ starting from $x = a$ and ending at $x = b$.*

Example 2: Calculate the definite integral of $f(x) = x^2$ on [0.2]. (This is Example 10 of Lecture 1 reformulated in the form of a definite integral).

Solution 2:

$$I = \int_0^2 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{2i}{n}\right)^2}_{f(x_i^*)} \underbrace{\left(\frac{2}{n}\right)}_{\Delta x_i} = \lim_{n\to\infty} \left(\frac{8}{n^3}\right) \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{4}{3} \frac{(n+1)(2n+1)}{n^2} = \frac{8}{3}.$$

Important remarks on the relation between an area and a definite integral:

- An area, defined as the physical measure of the size of a 2D domain, is always non-negative.
- The value of a definite integral, sometimes also referred to as an "area", can be both positive and negative.
- This is because: a definite integral = the limit of Riemann sums. But Riemann sums are defined as

$$R_n = \sum_{i=1}^{n} (\text{area of i}^{\text{th}} \text{ rectangle}) = \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{height} \underbrace{\Delta x_i}_{width}.$$

- Note that both the hieght $f(x_i^*)$ and the width Δx_i can be negative implying that R_n can have either signs.

3.3 The fundamental theorem of calculus

3.4 Areas between two curves

Example:

(i)

$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{de^x}{1+(e^x)^2} = \int \frac{du}{1+u^2} = tan^{-1}(u) + C = tan^{-1}(e^x) + C.$$

(ii)

$$\int \frac{x}{\sqrt{e^{2x^2}-1}}dx = \frac{1}{2}\int \frac{dx^2}{\sqrt{e^{2x^2}(1-e^{-2x^2})}} = \frac{1}{2}\int \frac{du}{\sqrt{e^{2u}(1-e^{-2u})}} = \frac{1}{2}\int \frac{du}{e^u\sqrt{1-(e^{-u})^2}}$$

$$= -\frac{1}{2}\int \frac{de^{-u}}{\sqrt{1-(e^{-u})^2}} = -\frac{1}{2}\int \frac{dy}{\sqrt{1-y^2}} = \frac{1}{2}cos^{-1}(y)+C = \frac{1}{2}cos^{-1}(e^{-u})+C = \frac{1}{2}cos^{-1}(e^{-x^2})+C.$$

Edwards/Penney $5^{th}-ed$ 9.2 Problems (difficult ones!).

(17) $\int \frac{e^{2x}}{1+e^{4x}}dy,\ (u=e^{2x})$ (19) $\int \frac{3x}{\sqrt{1-x^4}}dx,\ (u=x^2)$ (23) $\int \frac{cos\theta}{1+sin^2\theta}d\theta,\ (u = sin\theta)$

(27) $\int \frac{1}{(1+t^2)tan^{-1}t}dt,\ (u = tan^{-1}t)$ (29) $\int \frac{1}{\sqrt{e^{2x}-1}}dx,\ (u=e^{-x})$

3. Special Trigonometric Substitutions

Example:

(i)

$$\int \frac{x^3}{\sqrt{1-x^2}}dx = \int \frac{\sin^3 u d\sin u}{\sqrt{1-\sin^2 u}} = \int \sin^3 u du = -\int (1-\cos^2 u)d\cos u$$

$$= \frac{cos^3 u}{3} - \cos u + C = \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + C.$$

Edwards/Penney $5^{th}-ed$ 9.6 Problems (difficult ones!).

(1) $\int \frac{1}{\sqrt{16-x^2}}dx,\ (x = 4\sin u)$ (9) $\int \frac{\sqrt{x^2-1}}{x}dx,\ (x=\cosh u)$ (11) $\int x^3\sqrt{9+4x^2}dx,\ (2x = 3\sinh u)$

(13) $\int \frac{\sqrt{1-4x^2}}{x}dx,\ (2x = \sin u)$ (19) $\int \frac{x^2}{\sqrt{1+x^2}}dx,\ (x = \sinh u)$ (27) $\int \sqrt{9+16x^2}dx,\ (4x = 3\sinh u)$

11 Integration by Parts

Integration by Parts is the integral version of the Product Rule in differentiation. The Product Rule in terms of differentials reads,

$$d(uv) = vdu + udv.$$

Integrating both sides, we obtain

$$\int d(uv) = \int vdu + \int udv.$$

Note that $\int d(uv) = uv + C$, the above equation can be expressed in the following form,

$$\int udv = uv - \int vdu.$$

Generally speaking, we need to use Integration by Parts to solve many integrals that involve the product between two functions. In many cases, Integration by Parts is most efficient in solving integrals of the product between a polynomial and an exponential, a logarithmic, or a trigonometric function. It also applies to the product between exponential and trigonometric functions.

Example: $\int xe^x dx.$

Solution: In order to eliminate the power function x, we note that $(x)' = 1$. Thus,

$$\int xe^x dx = \int xde^x = xe^x - \int e^x dx = xe^x - e^x + C.$$

Example: $\int x^2 cosxdx.$

Solution: In order to eliminate the power function x^2, we note that $(x^2)'' = 2$. Thus, we need to use Integration by Patrs twice.

$$= \frac{1}{2}[x^2(\ln x)^2 - x^2 \ln x + \frac{x^2}{2}] + C = \frac{x^2}{2}[(\ln x)^2 - \ln x + \frac{1}{2}] + C.$$

More Exercises on Integration by Parts:

Example:

(i)

$$\int t \sin t dt = -\int t d\cos t = -[t\cos t - \int \cos t dt] = [\sin t - t\cos t] + C.$$

(ii)

$$\int tan^{-1}x dx = xtan^{-1}x - \int xdtan^{-1}x = xtan^{-1}x - \int \frac{x}{1+x^2}dx$$

$$= xtan^{-1}x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = xtan^{-1}x - \ln\sqrt{1+x^2} + C.$$

Edwards/Penney 5^{th} − ed. 9.3 Problems.

(1) $\int xe^{2x}dx$ (5) $\int x\cos(3x)dx$ (7) $\int x^3 \ln x dx,\ (x>0)$

(11) $\int \sqrt{y}\ln y dy$ (13) $\int (\ln t)^2 dt$ (19) $\int csc^3\theta d\theta$

(21) $\int x^2 tan^{-1}x dx$ (27) $\int xcsc^2 x dx$ (19) $\int csc^3\theta d\theta$

12 Integration by Partial Fractions

Rational functions are defined as the quotient between two polynomials:

$$R(x) = \frac{P_n(x)}{Q_m(x)}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m respectively. The method of partial fractions is an **algebraic** technique that decomposes $R(x)$ into a sum of terms:

Example: $I = \int \cos(\sqrt{x})dx$.

Solution: Note that $cos(\sqrt{x})$ is a composite fucntion. The substitution to change it into a simple, elementary function is $u = \sqrt{x}$, $du = u'dx = \frac{1}{2\sqrt{x}}dx$ or $dx = 2\sqrt{x}du = 2udu$. However, note that $x = (\sqrt{x})^2$,

$$I = \int \cos(\sqrt{x})dx = \int \cos(\sqrt{x})d(\sqrt{x})^2 = \int cos(u)du^2 = 2\int ucos(u)du$$

$$= 2\int udsin(u) = 2[usin(u) + cos(u)] + C = 2[\sqrt{x}sin(\sqrt{x}) + cos(\sqrt{x})] + C.$$

Example: $I = \int sec(x)dx = \int (1/cos(x))dx$.

Solution: This is a difficult integral. Note that $1/cos(x)$ is a composite fucntion. One substitution to change it into a simple, elementary function is $u = cos(x)$. However, $x = cos^{-1}(u)$ and $dx = x'du = \frac{-du}{\sqrt{1-u^2}}$. $1/\sqrt{1-u^2}$ is still a composite function that is difficult to deal with. Here is how we can deal with it. It involves trigonometric substitution followed by partial fractions.

$$I = \int \frac{dx}{cos(x)} = \int \frac{cos(x)dx}{cos^2(x)} = \int \frac{dsin(x)}{1 - sin^2(x)} = \int \frac{du}{1-u^2} = -\int [\frac{1}{u-1} - \frac{1}{u+1}]\frac{du}{2}$$

$$= \ln\sqrt{|\frac{u+1}{u-1}|} + C = \ln|\frac{u+1}{\sqrt{1-u^2}}| + C = \ln|\frac{sin(x)+1}{cos(x)}| + C = \ln|tan(x) + sec(x)| + C,$$

where $u = sin(x)$ and $\sqrt{1-u^2} = cos(x)$ were used!

More Exercises on Integration by Multiple techniques:

1. Substitution followed by Integration by Parts

Example:
(i)

$$\int x^3\sin(x^2)dx = \frac{1}{2}\int x^2\sin(x^2)dx^2 = \frac{1}{2}\int u\sin udu = \frac{-1}{2}\int ud\cos u$$

$$= \frac{1}{2}[\sin u - u\cos u] + C = \frac{1}{2}[\sin(x^2) - x^2\cos(x^2)] + C.$$

The formula is no longer valid if the density is NOT a constant but is nonuniformly distributed in the solid. However, if you cut the mass up into infinitely many pieces of inifinitely small volume dV, then the density in this little volume can be considered a constant so that we can apply the above formula. Thus, the infinitely small mass contained in the little volume is

$$dm = density \times (infinitely\ small\ volume) = \rho dV.$$

The total mass is obtained by adding up the masses of all such small pieces.

$$m = \int_0^{m(V(b))} dm = \int_{V(a)}^{V(b)} \rho dV,$$

where $V(x) = \int_0^{V(x)} dV$ is the volume of the portion of the solid between a and x; while $m(V(x)) = \int_0^{m(V(x))} dm$ is the mass contained in the volume $V(x)$.

Example: Calculate the total amount of pollutant in an exhaust pipe filled with polluted liquid which connects a factory to a river. The pipe is 100 m long with a diameter of 1 m. The density of the pollutant in the pipe is $\rho(x) = e^{-x/10}$ (kg/m^3), x is the distance in meters from the factory.

Solution: Since the density only varies as a function of the distance x, we can "cut" the pipe into thin cylindrical disks along the axis of the pipe. Thus, $dV = d(\pi r^2 x) = \pi r^2 dx$. Using the integral form of the formula, we obtain

$$m = \int_{m(0)}^{m(100)} dm = \int_{V(0)}^{V(100)} \rho(x) dV$$

$$= \int_0^{100} \rho(x) \pi r^2 dx = \frac{\pi}{4} \int_0^{100} e^{-x/10} dx = \frac{10\pi}{4}[1 - e^{-10}] \approx 7.85\ (kg).$$

Example: The air density h meters above the earth's surface is $\rho(h) = 1.28e^{-0.000124h}$ (kg/m^3). Find the mass of a cylindrical column of air 4 meters in diameter and 25 kilometers high. (3 points)

Solution: Similar to the previous problem, the density only varies as a function of the altitutde h. Thus, we "cut" this air column into horizontal slices of thin disks with volume $dV = \pi r^2 dh$.

The Advantage of Trigonometric Functions

Relations between different trigonometric functions are very important since when we differentiate or integrate one trig function we obtain another trig function. When we use trig substitution, it is often a necessity for us to know the definition of other trig functions that is related to the one we use in the substitution.

Example: Calculate the derivative of $y = \sin^{-1} x$.

Solution: $y = \sin^{-1} x$ means $\sin y = x$. Differentiate both sides of $\sin y = x$, we obtain

$$\cos y y' = 1 \ \Rightarrow \ y' = \frac{1}{\cos y}.$$

In order to express y' in terms of x, we need to express $\cos y$ in terms of x. Given that $\sin y = x$, we can obtain $\cos y = \sqrt{\cos^2 y} = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ by using trig identities. However, it is easier to construct a right triangle with an angle y. The opposite side must be x while the hypotenuse must be 1, thus the adjacent side is $\sqrt{1 - x^2}$. Therefore,

$$\cos y = \frac{adjacent\ side}{hypotenuse} = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

Example: Calculate the integral $\int \sqrt{1 + x^2} dx$.

Solution: This integral requires standard trig substitution $x = \tan u$ or $x = \sinh u$. Let's use $x = \sinh u$. Recall that $1 + \sinh^2 u = \cosh^2 u$ and that $dx = d \sinh u = \cosh u du$,

$$\int \sqrt{1 + x^2} dx = \int \sqrt{1 + \sinh^2 u} d \sinh u = \int \cosh^2 u du$$

$$= \frac{1}{2} \int [1 + \cosh(2u)] du = \frac{1}{2}[u + \frac{1}{2} \sinh(2u)] + C = \frac{1}{2}[u + \sinh u \cosh u] + C.$$

where hyperbolic identities $\cosh^2 u = [1 + \cosh(2u)]/2$ and $\sinh(2u) = 2 \sinh u \cosh u$ were used. However, we need to express the solution in terms of x. Since $x = \sinh u$ was the substitution, we know right away $u = \sinh^{-1} x = \ln |x + \sqrt{1 + x^2}|$ and $\sinh u = x$, but how to express $\cosh u$ in terms of x? We can solve it using hyperbolic identities. $\cosh u = \sqrt{\cosh^2 u} = \sqrt{1 + \sinh^2 u} = \sqrt{1 + x^2}$. Thus,

$$\frac{1}{2}[u + \sinh u \cosh u] = \frac{1}{2}[\ln |x + \sqrt{1 + x^2}| + x \sqrt{1 + x^2}].$$