Method of Undetermined Coefficients

by Andrew Binder

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Problem. Solve the initial value problem $y'' + 6y' + 13y = 9\cos(2t) - 87\sin(2t),\ y(0) = 9,\ y'(0) = -4$.

Solution. In order to solve this second order differential equation, we will use the method of undetermined coefficients. The first step of the method of undetermined coefficients is to find the homogeneous solution to the differential equation. We will call the homogeneous solution y_h. The homogeneous solution satisfies

$$y_h'' + 6y_h' + 13y_h = 0. \tag{1}$$

Notice that the above equation is simply the left hand side of our initial problem set equal to zero. We can solve this homogeneous equation using either the factoring operator method or the characteristic equation method. I will use the characteristic equation method. The characteristic equation for this second order differential equation is

$$r^2 + 6r + 13 = 0. \tag{2}$$

The roots of the characteristic equation may be found using the quadratic equation. The roots of the characteristic equation are $r = -3 \pm 2i$. The general homogeneous solution is then

$$y_h = C_1 e^{-3t}\cos(2t) + C_2 e^{-3t}\sin(2t), \tag{3}$$

where C_1 and C_2 are arbitrary constants. The second step in this method is to guess a particular solution of our given problem. A particular solution, which we will call y_p, satisfies

$$y_p'' + 6y_p' + 13y_p = 9\cos(2t) - 87\sin(2t). \tag{4}$$

Our guess for the particular solution depends on the functions on the right hand side of the above equation. On page 519 of the class text, you will find a list of common initial guesses for particular solutions as well as the general procedure for choosing which guess to use. In our case, the right hand side consists of trigonometric functions. A good guess for this scenario is

$$y_p = A\sin(2t) + B\cos(2t), \tag{5}$$

where A and B are undetermined constants. In order to determine A and B, we will substitute y_p into Eq. (4) and solve for them. Before I do so, let me calculate the derivatives of y_p that appear in Eq. (4):

$$\begin{aligned} y_p' &= 2A\cos(2t) - 2B\sin(2t) \\ y_p'' &= -4A\sin(2t) - 4B\cos(2t). \end{aligned}$$

Now, substitute y_p and its derivatives into Eq. (4):

$$[-4A\sin(2t) - 4B\cos(2t)] + 6[2A\cos(2t) - 2B\sin(2t)] + 13[A\sin(2t) + B\cos(2t)] = 9\cos(2t) - 87\sin(2t). \tag{6}$$

Simplify:

$$[9B + 12A]\cos(2t) + [9A - 12B]\sin(2t) = 9\cos(2t) - 87\sin(2t). \tag{7}$$

From here, we set $9B + 12A = 9$ and $9A - 12B = -87$, and solve the system of linear equations. To get to that point, I set the coefficient of the sine function on the left hand side equal to the coefficient of the sine function on the right hand side. A similar procedure was used for the cosine terms. Solving this sytem of linear equations, you will find that $A = -3$ and $B = 5$. Therefore, our particular solution is

$$y_p = -3\sin(2t) + 5\cos(2t). \tag{8}$$