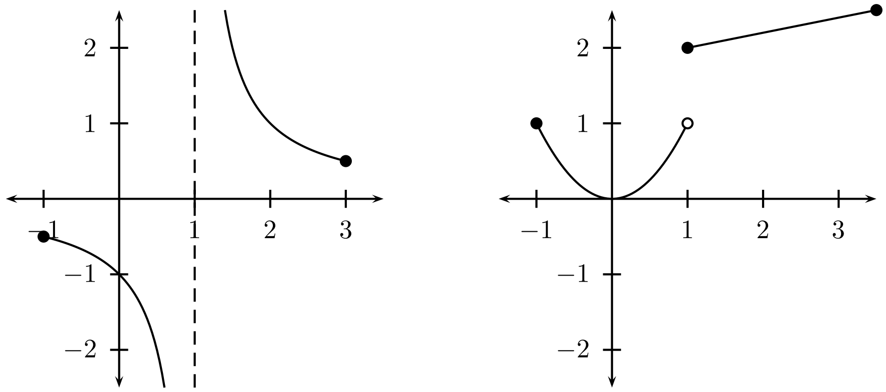

$$f(x) = \frac{1}{x-1}$$
$$x \in [-1,3] - \{1\}$$

$$f(x) = \begin{cases} x^2, & -1 \le x < 1 \\ 2 + 0.2(x-1), & 1 \le x \le 3.5 \end{cases}$$

Figure 3.4: Theorems 3.3.1 and 3.3.2—and their corollaries EVT and IVT—require continuity of f, without which the image can be disconnected or unbounded. In the first graph, there is a vertical asymptote at $x = 1$, which is a point of discontinuity. The image $(-\infty, -1/2] \cup [1/2, \infty)$ is disconnected (so no IVT conclusion) and unbounded (so no EVT conclusion). In the second case, there is a "jump" discontinuity at $x = 1$, and the image $[0,1] \cup [2, 2.5]$ is also disconnected (no IVT conclusion), though bounded. It happens that the second graph does have maximum and minimum values $f(0) = 0$ and $f(3.5) = 2.5$ though this was not guaranteed because $f(x)$ is not continuous on all of $[-1, 3.5]$. These examples do not violate the corollaries IVT and EVT since both corollaries claim the truth of tautologies of the form $P \longrightarrow Q$, which is equivalent to $(\sim P) \vee Q$, and here we have $\sim P$ in both cases, making them true to the theorem and corollaries vacuously (in form $P \to Q$) or trivially (using the form $(\sim P) \vee Q$).

or minimum is actually achieved; in the third graph, the image is unbounded from above so no maximum is achieved, and no minimum is achieved either. It may happen that $f((a,b))$ is a closed and bounded interval, as in the second graph in that figure, but it clearly (from the other two graphs in that figure) is not guaranteed. Continuity is also required in these theorems, as we see in Figure 3.4.

Note that the first function in the figure is continuous on $[-1,1)$ because it is continuous on $(-1,1)$ and right-continuous at -1. Similarly it is continuous on $(1,3]$. The second function is continuous on $[-1,1)$ and $[1,3.5]$. That is **not** to say it is continous at each $x \in [1,3.5]$, but rather that the "piece" drawn on that interval is a continuous "piece," in the sense of Definition 3.3.2, page 195 and the discussion following that.

3.3.2 Simple Applications of the Intermediate Value Theorem

We will return to the extreme value theorem and its applications later in the text. Here we will instead look at the IVP and its usefulness in algebra. The following simple theorem can often be useful when we look at continuity considerations:

Theorem 3.3.3 *If I and J are intervals of any kind except for single points, with $I \subseteq J$, and $f : J \longrightarrow \mathbb{R}$ is continuous on J, then $f : I \longrightarrow \mathbb{R}$ is continuous on I.*

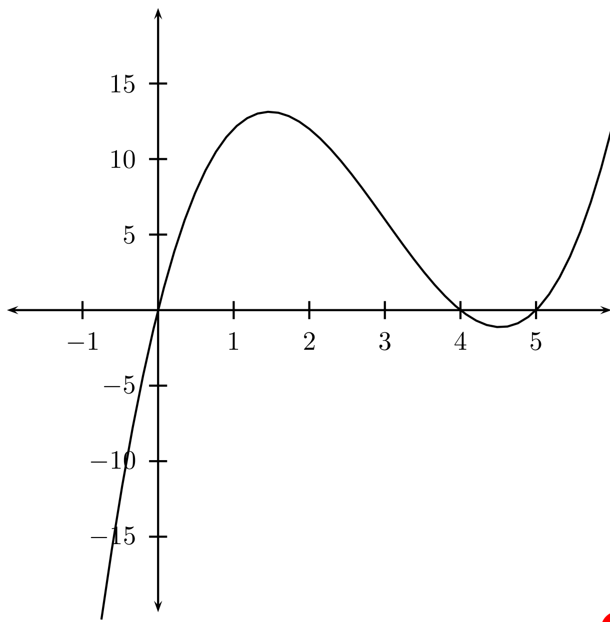

Figure 3.5: Actual graph of $f(x) = x(x-4)(x-5)$, showing where $f(x) > 0$, where $f(x) < 0$, and how the transitions between these occur, in this case by the function's height passing through zero. Compare to the sign chart in Example 3.3.3.

Example 3.3.3 was relatively straightforward. There can be complications, and we have to be careful to answer the given question. For instance, we do not always have *strict inequalities* $<$, $>$, but may have *inclusive inequalities* $\leq$, $\geq$.

Example 3.3.4 *Solve $x^2 \geq x + 1$.*

Solution: First we subtract, and then define $f(x) = x^2 - x - 1$, so that

$$x^2 \geq x + 1 \iff x^2 - x - 1 \geq 0 \iff f(x) \geq 0.$$

Now solving $f(x) = 0$ requires the quadratic formula or completing the square. We will opt for the former. Recall first that $f(x) = 0 \iff x^2 - x - 1 = 0$.

$$f(x) = 0 \iff x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5} \approx -0.61803,\ 1.61803.$$

We will always use the exact values, but the approximate ones are also useful since we need to know where to find our test points.[17]

[17] We could factor $f(x)$ based upon the solutions to $f(x) = 0$, namely $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$:

$$f(x) = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right).$$

Such an approach is perhaps more sophisticated than our method in Example 3.3.4, where we did not bother to factor $f(x)$, but is often unwieldy and requires more subtlety than necessary to solve the inequality.