by $\text{Dom}(g(f)) \equiv \{x \in \text{Dom}(f) : f(x) \in \text{Dom}(g)\}$. We now let a be a limit point of $\text{Dom}(g(f))$ and consider two situations. First, if b is a number such that

$$\lim_{x \to a} f(x) = b\,, \qquad \text{and} \qquad \lim_{y \to b} g(y) = g(b)\,, \tag{1.8}$$

then we have the **composition limit rule:**

$$\lim_{x \to a} g(f(x)) = g\Big(\lim_{x \to a} f(x)\Big) = g(b)\,. \tag{1.9}$$

Second, if

$$\lim_{x \to a} f(x) = f(a)\,, \tag{1.10}$$

and f is either increasing or decreasing over an open interval containing a, then we have the **change of variable limit rule:**

$$\lim_{y \to f(a)} g(y) = \lim_{x \to a} g(f(x))\,. \tag{1.11}$$

Such a simple rules do not generally hold for one-sided limits.

1.4: Continuity. A function f is said to be **continuous** at a point a in $\text{Dom}(f)$ if either a is not a limit point of $\text{Dom}(f)$ or

$$\lim_{x \to a} f(x) = f(a)\,. \tag{1.12}$$

Here (1.12) is asserting two things:

- the limit on the left side of (1.12) exists;
- the limit equals $f(a)$.

You should know examples of functions that fail to be continuous at a point in its domain both where the limit on the left of (1.12) fails to exist and where the limit exists but does not equal $f(a)$. You should be able to tell by looking at the graph of a function where it is continuous.

It follows from the sum, product and quotient limit rules (1.7) that if f and g are functions that are both continuous at the point a then the functions $f \pm g$ and $f\,g$ will be continuous at the point a, as will the function f/g provided $g(a) \neq 0$. Moreover, the composition limit rule (1.9) shows that if f continuous at the point a while g is continuous at the point $f(a)$ then the composition $g(f)$ is continuous at the point a.

A function that is continuous at every point in an interval is said to be continuous over that interval. Roughly speaking, when drawing the graph of such a function f over such an interval, one need not lift the pen or pencil from the paper. This is because (1.12) states that as the pen moves along the graph $(x, f(x))$ it will approach the point $(a, f(a))$ as x tends to a. The graph of f will consequently have no breaks, jumps, or holes over the interval.

A function that is continuous at every point in its domain is said to be continuous. Every elementary function is continuous.

2.4: Basic Derivatives from the Definition. There are a few basic functions whose derivative formulas you should be able to derive directly from the definition (2.7). These include

$$
\begin{aligned}
&\frac{d}{dx}\,1 = 0\,, && \frac{d}{dx}\,x^n = nx^{n-1}\,, \\
&\frac{d}{dx}\,e^x = e^x\,, && \frac{d}{dx}\,\ln(x) = \frac{1}{x}\,, \\
&\frac{d}{dx}\,a^x = \ln(a)a^x\,, && \frac{d}{dx}\,\log_a(x) = \frac{1}{\ln(a)}\,\frac{1}{x}\,, \\
&\frac{d}{dx}\,\sin(x) = \cos(x)\,, && \frac{d}{dx}\,\cos(x) = -\sin(x)\,, \\
&\frac{d}{dx}\,\sinh(x) = \cosh(x)\,, && \frac{d}{dx}\,\cosh(x) = \sinh(x)\,,
\end{aligned}
\tag{2.11}
$$

and any simple variants thereof. The top two are straightforward. The first is trivial, and when n is an integer the second only requires simple algebraic manipulation of the difference quotient before passing to the limit. For example, when n is a positive integer you have to expand $(x+h)^n$ by the binomial formula. You should be comfortable with cases in which n is a positive or negative integer whose absolute value is not too large.

The formulas for the expontential and logarithmic derivatives are derived using the fact, which you should know, that the number e is given by the limit

$$e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}\,. \tag{2.12}$$

Given this limit, you should be able to obtain the derivative formulas for logarithms. You should also be able to use (2.12) and the change of variable limit rule (1.10) with $s = a^h - 1$ to derive the limit

$$\lim_{h\to 0}\frac{a^h - 1}{h} = \ln(a)\,. \tag{2.13}$$

From this you should be able to obtain derivative formulas for the exponentials.

The formulas for the sine and cosine derivatives are obtained through the appropriate trigonometric addition formulas and the limits

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1\,, \qquad \lim_{h\to 0}\frac{1-\cos(h)}{h} = 0\,. \tag{2.14}$$

The first limit was argued in class by comparing the area of a pizza pie slice of angle h with that of a larger and a smaller triangle. This led us to the inequalities

$$\cos(h) \le \frac{\sin(h)}{h} \le 1 \qquad \text{for every } |h| < \tfrac{\pi}{2}\,. \tag{2.15}$$

Given this inequality, it is easy to obtain the first limit. Given the first limit, you should be able to obtain the second limit. Given both limits, you should be able to derive the sine and cosine derivative formulas.

minus the top times the derivative of the bottom, all over the bottom squared", or more poetically, "bottom-dee-top minus top-dee-bottom over bottom squared". While it is very helpful to have this rule memorized, it is not critical. In every instance that the quotient rule can be applied, the quotient can be recast as a product to which the product rule (3.4) can be applied. That is after all how the quotient rule was derived above.

If the general Leibnitz rule (3.5) is specialized to the case where all the functions u_k are the same function u then it reduces to the **monomial power rule:**

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx} \,. \tag{3.8}$$

The monomial power rule was derived above for positive integers n. When it is combined with the reciprocal rule (3.6), one sees that it extends to negative integers n. This rule can be extended further. Namely, given any differentiable function u and any rational number p for which u^p is defined, the function u^p is differentiable wherever u^{p-1} is defined and its derivative is given by the **rational power rule:**

$$\frac{d}{dx} u^p = p u^{p-1} \frac{du}{dx} \,. \tag{3.9}$$

Wherever $u \neq 0$ this rule can be derived as follows. Because p is rational it can be expressed as $p = m/n$ where m and n are integers and $n > 0$. If the monomial power rule (3.8) is then applied to each side of the identity $(u^p)^n = u^m$, one finds that

$$n (u^p)^{n-1} \frac{d}{dx} u^p = m u^{m-1} \frac{du}{dx} \,,$$

which is equivalent to the rational power rule wherever $u \neq 0$. Points where $u = 0$ and $p \geq 1$ can be treated directly from the definition of the derivative.

3.3: Rules for Compositions of Functions. Given two differentiable functions v and u, the derivative of their composition $v(u)$ is given by the **chain rule:**

$$\frac{d}{dx} v(u) = v'(u) \, \frac{du}{dx} \,. \tag{3.10}$$

This is also tricky to express in words, but may be rendered as "the derivative of a composition is the derivative of the outer, evaluated at the inner, times the derivative of the inner". You could also say "the derivative of a composition is the product of the derivatives", provided you realize that this leaves a lot unsaid about the arguments of the derivatives involved. If the functions u and v relate the variables x, y, and z by $z = v(y)$ and $y = u(x)$, then (3.10) may be expressed as

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} \,. \tag{3.11}$$