At first this looks useless—we're right back to $\int \sec^3 x\,dx$. But looking more closely:

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
\int \sec^3 x\,dx + \int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx \\
2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx \\
\int \sec^3 x\,dx &= \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx \\
&= \frac{\sec x \tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C.
\end{aligned}$$

□

EXAMPLE 8.4.4 Evaluate $\int x^2 \sin x\,dx$. Let $u = x^2$, $dv = \sin x\,dx$; then $du = 2x\,dx$ and $v = -\cos x$. Now $\int x^2 \sin x\,dx = -x^2 \cos x + \int 2x \cos x\,dx$. This is better than the original integral, but we need to do integration by parts again. Let $u = 2x$, $dv = \cos x\,dx$; then $du = 2$ and $v = \sin x$, and

$$\begin{aligned}
\int x^2 \sin x\,dx &= -x^2 \cos x + \int 2x \cos x\,dx \\
&= -x^2 \cos x + 2x \sin x - \int 2 \sin x\,dx \\
&= -x^2 \cos x + 2x \sin x + 2\cos x + C.
\end{aligned}$$

□

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

sign	u	dv
	x^2	$\sin x$
$-$	$2x$	$-\cos x$
	2	$-\sin x$
$-$	0	$\cos x$

or

u	dv
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

7. $\int x \arctan x \, dx \Rightarrow$

8. $\int x^3 \sin x \, dx \Rightarrow$

9. $\int x^3 \cos x \, dx \Rightarrow$

10. $\int x \sin^2 x \, dx \Rightarrow$

11. $\int x \sin x \cos x \, dx \Rightarrow$

12. $\int \arctan(\sqrt{x}) \, dx \Rightarrow$

13. $\int \sin(\sqrt{x}) \, dx \Rightarrow$

14. $\int \sec^2 x \csc^2 x \, dx \Rightarrow$

8.5 Rational Functions

A **rational function** is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2+x-6}, \qquad \frac{1}{(x-3)^2}, \qquad \frac{x^2+1}{x^2-1},$$

are all rational functions of x. There is a general technique called "partial fractions" that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial ax^2+bx+c.

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form $(ax+b)^n$, the substitution $u = ax+b$ will always work. The denominator becomes u^n, and each x in the numerator is replaced by $(u-b)/a$, and $dx = du/a$. While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.

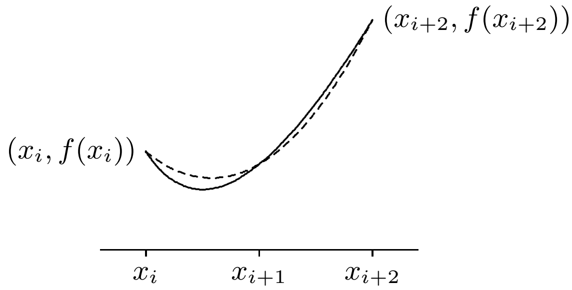

Figure 8.6.3 A parabola (dashed) approximating a curve (solid).

THEOREM 8.6.3 Suppose f has a fourth derivative $f^{(4)}$ everywhere on the interval $[a, b]$, and $|f^{(4)}(x)| \le M$ for all x in the interval. With $\Delta x = (b-a)/n$, an error estimate for Simpson's approximation is

$$E(\Delta x) = \frac{b-a}{180} M(\Delta x)^4 = \frac{(b-a)^5}{180n^4} M.$$

■

EXAMPLE 8.6.4 Let us again approximate $\int_0^1 e^{-x^2}\,dx$ to two decimal places. The fourth derivative of $f = e^{-x^2}$ is $(16x^4 - 48x^2 + 12)e^{-x^2}$; on $[0, 1]$ this is at most 12 in absolute value. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$, but taking a cue from our earlier example, let's require $E(\Delta x) < 0.001$:

$$\begin{aligned} \frac{1}{180}(12)\frac{1}{n^4} &< 0.001 \\ \frac{200}{3} &< n^4 \\ 2.86 \approx \sqrt[4]{\frac{200}{3}} &< n \end{aligned}$$

So we try $n = 4$, since we need an even number of subintervals. Then the error estimate is $12/180/4^4 < 0.0003$ and the approximation is

$$(f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1))\frac{1}{3 \cdot 4} \approx 0.746855.$$

So the true value of the integral is between $0.746855 - 0.0003 = 0.746555$ and $0.746855 + 0.0003 = 0.7471555$, both of which round to 0.75. □

13. $\displaystyle\int \frac{1}{t^2+3t}\,dt \Rightarrow$

14. $\displaystyle\int \frac{1}{t^2\sqrt{1+t^2}}\,dt \Rightarrow$

15. $\displaystyle\int \frac{\sec^2 t}{(1+\tan t)^3}\,dt \Rightarrow$

16. $\displaystyle\int t^3\sqrt{t^2+1}\,dt \Rightarrow$

17. $\displaystyle\int e^t \sin t\,dt \Rightarrow$

18. $\displaystyle\int (t^{3/2}+47)^3\sqrt{t}\,dt \Rightarrow$

19. $\displaystyle\int \frac{t^3}{(2-t^2)^{5/2}}\,dt \Rightarrow$

20. $\displaystyle\int \frac{1}{t(9+4t^2)}\,dt \Rightarrow$

21. $\displaystyle\int \frac{\arctan 2t}{1+4t^2}\,dt \Rightarrow$

22. $\displaystyle\int \frac{t}{t^2+2t-3}\,dt \Rightarrow$

23. $\displaystyle\int \sin^3 t\cos^4 t\,dt \Rightarrow$

24. $\displaystyle\int \frac{1}{t^2-6t+9}\,dt \Rightarrow$

25. $\displaystyle\int \frac{1}{t(\ln t)^2}\,dt \Rightarrow$

26. $\displaystyle\int t(\ln t)^2\,dt \Rightarrow$

27. $\displaystyle\int t^3 e^t\,dt \Rightarrow$

28. $\displaystyle\int \frac{t+1}{t^2+t-1}\,dt \Rightarrow$