THE CARTESIAN PLANE (CONTINUED)

a measure of how fast the line moves "up" for every bit that it moves "over" (left or right). If (a,b) and (c,d) are two points on the line, then the slope is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{d-b}{c-a}.$$

—Any pair of points on a straight line will give the same slope value.

—**Horizontal lines** have slope 0.

—The slope of a **vertical line** is **undefined;** it is "infinitely large."

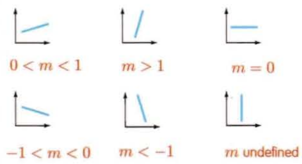
—Lines that go "up right" and "down left" (ending in I and III) have **positive slope.**

—Lines that go "up left" and "down right" (ending in II and IV) have **negative slope.**

—**Parallel lines** have the same slope.

—The slopes of **perpendicular lines** are negative reciprocals of each other: if two lines of slope m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$ and $m_2 = -\frac{1}{m_1}$.

Rough direction of lines with slope m:



***x*-Intercept:** The x-coordinate of the point where a line crosses the x-axis. The x-intercept of a line that crosses the x-axis at $(a,0)$ is a. Horizontal lines have no x-intercept.

***y*-Intercept:** The y-coordinate of the point where a line crosses the y-axis. The y-intercept of a line that crosses the y-axis at $(0,b)$ is b. Vertical lines have no y-intercept.

FINDING THE EQUATION OF A LINE

Any line in the Cartesian plane represents some linear relationship between x and y values. The relationship always can be expressed as $Ax + By = C$ for some real numbers A, B, C. The coordinates of every point on the line will satisfy the equation.

A **horizontal line** at height b has equation $y = b$.

A **vertical line** with x-intercept a has equation $x = a$.

Given slope m and y-intercept b:

Equation: $y = mx + b$.

Standard form: $mx - y = -b$.

Given slope m and any point (x_0, y_0):

Equation: $y - y_0 = m(x - x_0)$.

Standard form: $mx - y = mx_0 - y_0$.

Alternatively, write down $y_0 = mx_0 + b$ and solve for $b = y_0 - mx_0$ to get the slope-intercept form.

Given two points (x_1, y_1) and (x_2, y_2):

Find the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Equation:

$y - y_1 = m(x - x_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, or

$y - y_2 = m(x - x_2)$.

Given slope m and x-intercept a:

Equation: $x = \frac{y}{m} + a$.

Given x-intercept a and y-intercept b:

Equation: $\frac{x}{a} + \frac{y}{b} = 1$.

Given a point on the line and the equation of a parallel line:

Find the slope of the parallel line (see Graphing Linear Equations). The slope of the original line is the same. Use point-slope form.

Given a point on the line and the equation of a perpendicular line:

Find the slope m_0 of the perpendicular line. The slope of the original line is $-\frac{1}{m_0}$. Use point-slope form.

GRAPHING LINEAR EQUATIONS

A **linear equation in two variables** (say x and y) can be manipulated—after all the x-terms and y-terms and constant terms are have been grouped together—into the form $Ax + By = C$. The graph of the equation is a straight line (hence the name).

—**Using the slope to graph:** Plot one point of the line. If the slope is expressed as a ratio of small whole numbers $\pm\frac{r}{s}$, keep plotting points r up and $\pm s$ over from the previous point until you have enough to draw the line.

—**Finding intercepts:** To find the y-intercept, set $x = 0$ and solve for y. To find the x-intercept, set $y = 0$ and solve for x.

SLOPE-INTERCEPT FORM: $y = mx + b$

One of the easiest-to-graph forms of a linear equation.

m is the slope.

b is the y-intercept. $(0, b)$ is a point on the line.

POINT-SLOPE FORM: $y - k = m(x - h)$

m is the slope.

(h, k) is a point on the line.

STANDARD FORM: $Ax + Bx = C$

Less thinking: Solve for y and put the equation into slope-intercept form.

Less work: Find the x- and the y-intercepts. Plot them and connect the line.

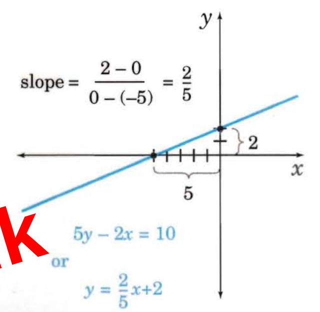
Slope: $-\frac{A}{B}$.

y-intercept: $\frac{C}{B}$.

x-intercept: $\frac{C}{A}$.

ANY OTHER FORM

If you are sure that an equation is linear, but it isn't in a nice form, find a couple of solutions. Plot those points. Connect them with a straight line. Done.



SIMULTANEOUS LINEAR EQUATIONS

A linear equation in two variables (say $ax + by = c$, with a and c not both zero) has infinitely many ordered pair (x, y) **solutions**—real values of x and y that make the equation true. Two simultaneous linear equations in two variables will have:

—Exactly **one solution** if their graphs intersect—the most common scenario.

—**No solutions** if the graphs of the two equations are parallel.

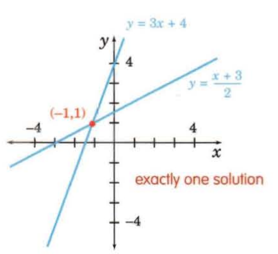
—**Infinitely many solutions** if their graphs coincide.

SOLVING BY GRAPHING: TWO VARIABLES

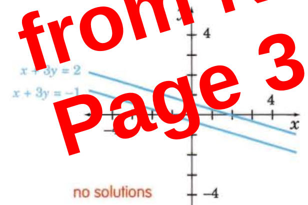
Graph both equations on the same Cartesian plane. The intersection of the graph gives the simultaneous solutions. (Since points on each graph correspond to solutions to the appropriate equation, points on *both graphs* are solutions to both equations.)

—Sometimes, the exact solution can be determined from the graph; other times the graph gives an estimate only. Plug in and check.

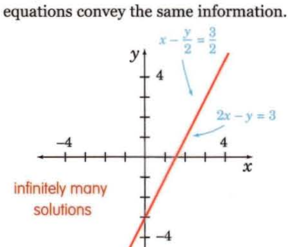
—If the lines intersect in exactly one point (most cases), the intersection is the unique solution to the system.



—If the lines are parallel, they do not intersect; the system has no solutions. *Parallel lines have the same slope;* if the slope is not the same, the lines will intersect.



—If the lines coincide, there are infinitely many solutions. Effectively, the two equations convey the same information.



SOLVING BY SUBSTITUTION: TWO VARIABLES

—Use one equation to solve for one variable (say, y) in terms of the other (x): isolate y on one side of the equation.

—Plug the expression for y into the other equation.

—Solve the resulting one-variable linear equation for x.

—If there is no solution to this new equation, there are no solutions to the system.

—If all real numbers are solutions to the new equation, there are infinitely many solutions; the two equations are **dependent**.

—Solve for y by plugging the x-value into the expression for y in terms of x.

—Check that the solution works by plugging it into the original equations.

Ex: $\begin{cases} x - 4y = 1 \\ 2x - 11 = 2y \end{cases}$

Using the first equation to solve for y in terms of x gives $y = \frac{1}{4}(x-1)$. Plugging in to the second equation gives $2x - 11 = 2\left(\frac{1}{4}(x-1)\right)$. Solving for x gives $x = 7$. Plugging in for y gives $y = \frac{1}{4}(7-1) = \frac{3}{2}$. Check that $(7, \frac{3}{2})$ works.

SOLVING BY ADDING OR SUBTRACTING EQUATIONS: TWO VARIABLES

Express both equations in the same form. $ax + by = c$ works well.

Look for ways to add or subtract the equations to eliminate one of the variables.

—If the coefficients on a variable in the two equations are the same, subtract the equations.

—If the coefficients on a variable in the two equations differ by a sign, add the equations.

—If one of the coefficients on one of the variables (say, x) in one of the equations is 1, multiply that whole equation by the x-coefficient in the other equation; subtract the two equations.

—If no simple combination is obvious, simply pick a variable (say, x). Multiply the first equation by the x-coefficient of the second equation, multiply the second equation by the x-coefficient of the first equation, and subtract the equations.

If all went well, the sum or difference equation is in one variable (and easy to solve if the original equations had been in $ax + by = c$ form). Solve it.

—If by eliminating one variable, the other is eliminated too, then there is no unique solution to the system. If there are no solutions to the sum (or difference) equation, there is no solution to the system. If all real numbers are solutions to the sum (or difference) equation, then the two original equations are dependent and express the same relationship between the variables; there are infinitely many solutions to the system.

—Plug the solved-for variable into one of the original equations to solve for the other variable.

Ex: $\begin{cases} x - 4y = 1 \\ 2x - 11 = 2y \end{cases}$

Rewrite to get $\begin{cases} x - 4y = 1 \\ 2x - 2y = 11 \end{cases}$

The x-coefficient in the first equation is 1, so we multiply the first equation by 2 to get $2x - 8y = 2$, and subtract this equation from the original second equation to get:

$(2-2)x + (-2-(-8))y = 11 - 2$ or $6y = 9$, which gives $y = \frac{3}{2}$, as before.

CRAMER'S RULE

The solution to the simultaneous equations

$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ is given by $\begin{array}{l} x = \frac{de - bf}{ad - bc} \\ y = \frac{af - ce}{ad - bc} \end{array}$

if $ad - bc \neq 0$.

MORE THAN TWO VARIABLES

There is a decent chance that a system of linear equations has a unique solution only if there are as many equations as variables.

—If there are too many equations, then the conditions are likely to be too restrictive, resulting in no solutions. (This is only actually true if the equations are "independent"—each new equation provides new information about the relationship of the variables.)

—If there are too few equations, then there will be too few restrictions; if the equations are not contradictory, there will be infinitely many solutions.

—All of the above methods can, in theory, be used to solve systems of more than two linear equations. In practice, graphing only works in two dimensions. It's too hard to visualize planes in space.

—Substitution works fine for three variables; it becomes cumbersome with more variables.

—Adding or subtracting equations (or rather, arrays of coefficients called **matrices**) is the method that is used for large systems.

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