LIMITS AND CONTINUITY (CONTINUED)

THE EPSILON–DELTA ($\varepsilon - \delta$) DEFINITIONS OF LIMITS AND CONTINUITY

Limits: $\lim_{x\to a} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever x is within δ of a, $f(x)$ is within ε of L (that is, $|x-a| < \delta$ implies that $|f(x) - L| < \varepsilon$).

Continuity: Function $f(x)$ is said to be continuous at $x = a$ if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever x_0 is within δ of a, $f(x_0)$ is within ε of $f(a)$ (that is, $|x_0 - a| < \delta$ implies that $|f(x_0) - f(a)| < \varepsilon$).

Equivalently, $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$.

LIMIT LAWS

Suppose $f(x)$ and $g(x)$ are two functions, a is a point (possibly $\pm\infty$) near which both $f(x)$ and $g(x)$ are defined. (These are only true if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and at least one of them is finite!)

Sum: $\lim\limits_{x\to a}\big(f(x) \pm g(x)\big) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

Scalar multiple: $\lim\limits_{x\to a}\big(cf(x)\big) = c\lim\limits_{x\to a} f(x)$ Here, c is any real.

Product: $\lim\limits_{x\to a} f(x)g(x) = \Big(\lim\limits_{x\to a} f(x)\Big)\Big(\lim\limits_{x\to a} g(x)\Big)$

Quotient: If $\lim\limits_{x\to a} g(x) \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

The Squeeze Theorem:
If $f(x) \le g(x) \le h(x)$ near $x = a$, and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x)$ exists and is equal to L.
The classic application of this theorem establishes that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Frequently-encountered limits:

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

$\lim\limits_{x\to 0} \frac{\cos x - 1}{x} = 0$

$\lim\limits_{x\to\infty} x^n e^{-x} = 0$ for all n

TAKING DERIVATIVES

INTUITION

If a and b are two points in the domain of $f(x)$ then the **average rate of change** of $f(x)$ on the interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$, a measure of how fast $f(x)$ has increased or decreased over the interval. This is also the slope of the line through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ on the graph of $f(x)$.

The derivative of $f(x)$ at a point $x = a$ is the **instantaneous rate of change**, a measure of how fast $f(x)$ is increasing or decreasing at a. Equivalently, the derivative is the **slope of the tangent line** to the graph of $f(x)$ at the point $x = a$—the unique line through the point $\big(a, f(a)\big)$ that touches the graph at only that point near $x = a$.

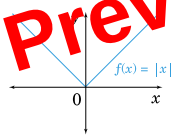
We compute the derivative $f'(a)$ by looking at the average rate of change of $f(x)$ on the interval $[a, a+h]$ and taking the limit as h goes to 0. Equivalently, $f'(a)$ is the limit as $h \to 0$ of the slope of the line through $\big(a, f(a)\big)$ and $\big(a+h, f(a+h)\big)$.

DEFINITION

If the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, we say that $f(x)$ is **differentiable at** $x = a$ and the limit is the **derivative of** $f(x)$ **at** $x = a$, denoted by $f'(a)$.

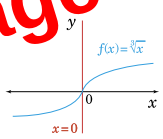
The function $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is the derivative function of $f(x)$. If it is defined whenever $f(x)$ is defined, then $f(x)$ is called **differentiable**.

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at a. The converse is not true: a function can be continuous but not differentiable. There are two cases where this occurs:

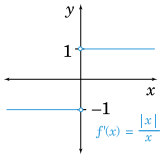


No tangent at $x = 0$

1. No tangent: **Ex:** $f(x) = |x|$ is continuous function is continuous at $x = 0$ since $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x| = 0$, but the derivative $f'(0)$ is undefined since the left-hand slope limit, $\lim_{h\to 0^-} \frac{|h|}{h} = -1$, does not equal the right-hand slope limit, $\lim_{h\to 0^+} \frac{|h|}{h} = 1$.

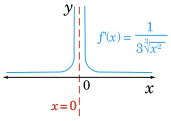


Vertical tangent at $x = 0$



$f'(0)$ undefined

2. Vertical tangent: The slope of a vertical line is "undefined." If $f(x)$ has a vertical tangent at $x = a$, then the derivative $f'(a)$ is undefined and the graph of $f'(x)$ will have a vertical asymptote at $x = a$. **Ex:** $f(x) = \sqrt[3]{x}$ has a vertical tangent at the point $(0, 0)$. The derivative function, $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$, goes to infinity at 0.



$f'(0)$ undefined

NOTATION

Different notations for the derivative function are useful in different contexts. The most common notations in calculus are $f'(x)$, y', $\frac{d}{dx}f(x)$, and $\frac{dy}{dx}$. The last two are in **Leibniz notation;** $\frac{dy}{dx}$ evolved from $\frac{\triangle y}{\triangle x} = \frac{\text{change in } y}{\text{change in } x}$, or slope. The expressions dy and dx represent infinitesmal changes in y and x.

- Higher-order derivatives can be written in "prime" notation: $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, or in Leibniz notation: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$.
- The derivative at a particular point a is most often expressed as $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.

METHODS AND TRICKS

Assume that $f(x)$ and $g(x)$ are two differentiable functions.

Sum and Difference: $\dfrac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$

Scalar Multiple: $\dfrac{d}{dx}\big(cf(x)\big) = cf'(x)$

Product: $\dfrac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$

MNEMONIC: If f is "hi" and g is "ho," then the product rule is "ho d hi plus hi d ho."

Quotient: $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

MNEMONIC: "Ho d hi minus hi d ho over ho ho."

The **Chain Rule** makes the derivative of composite functions. Here are two ways of writing it:

1. $\dfrac{d}{dx}\big(f(g(x))\big) = f'\big(g(x)\big)\,g'(x)$.
2. If $z = g(x)$ and $y = f(u) = f\big(g(x)\big)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

IMPLICIT DIFFERENTIATION

Implicit differentiation uses the product and chain rules to find slopes of curves when it is difficult or impossible to express y as a function of x. Leibniz notation may be easiest when differentiating implicitly. Take the derivative of each term in the equation with respect to x. Then rewrite $\frac{dy}{dx} = y'$ and $\frac{dx}{dx} = 1$ and solve for y'.

Ex 1: $x^2 + y^2 = 1$
Implicitly differentiating with respect to x gives the expression $2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$, which simplifies to $2x + 2yy' = 0$ or $y' = -\frac{x}{y}$. The derivative can now be found for any point on the curve, even though it is not actually a function. You will get the same result if you first solve for $y = \pm\sqrt{1-x^2}$ and keep track of the $\pm$ signs in different quadrants.

Ex 2: $x\cos y - y^2 = 3x$
Differentiate to obtain first $\frac{dx}{dx}\cos y + x\frac{d(\cos y)}{dx} - 2y\frac{dy}{dx} = 3\frac{dx}{dx}$, and then $\cos y - x\sin y y' - 2yy' = 3$. Finally, solve for $y' = \frac{\cos y - 3}{x\sin y + 2y}$.

DERIVATIVES OF BASIC FUNCTIONS

Constants:	$\dfrac{d(c)}{dx} = 0$	A constant function is always flat.
Linear:	$\dfrac{d(mx+b)}{dx} = m$	The line $y = mx + b$ has slope m.
Powers:	$\dfrac{d(x^n)}{dx} = nx^{n-1}$	True for all real $n \neq 0$.
Polynomial:	$\dfrac{d(a_nx^n + \cdots + a_2x^2 + a_1x + a_0)}{dx} = na_nx^{n-1} + \cdots + 2a_2x + a_1$	
Exponential:	$\dfrac{d(e^x)}{dx} = e^x$	This is why e is called the "natural" logarithm base: Ae^x are the only functions that are their own derivatives.
	$\dfrac{d(a^x)}{dx} = a^x \ln a$	When in doubt, convert a^x to $e^{x\ln a}$.
Logarithmic:	$\dfrac{d(\ln x)}{dx} = \dfrac{1}{x}$	Found using implicit differentiation.
	$\dfrac{d(\log_a x)}{dx} = \dfrac{1}{x\ln a}$	When in doubt, convert $\log_a x$ to $\frac{\ln x}{\ln a}$.

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