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CALCULUS II

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OVERVIEW

Calculus is the study of "nice"—smoothly changing—functions.

- **Differential calculus** studies how quickly a function is changing at a particular point. For more on differential calculus, see the *Calculus I* SparkChart.
- **Integral calculus** studies areas enclosed by curves and is used to compute a continuous (as opposed to discrete) summation. Integration is used in geometry to find the length of an arc, the area of a surface, and the volume of a solid; in physics, to compute the total work done by a varying force or the location of the center of mass of an irregular object; in statistics, to work with varying probabilities.
- **Differential equations** (diff-eqs) express a relationship between a function and its derivatives. Diff-eqs come up when modeling natural phenomena.
- **Infinite series** are special types of functions. Familiar functions can often be represented as infinite **Taylor polynomials.** Infinite series are used to differentiate and integrate difficult functions, as well as to approximate values of functions and their derivatives.

REVIEW OF TERMS

- A **function** is a rule that assigns to each value of the **domain** a unique value of the **range**.
- Function $f(x)$ is **continuous** on some interval if whenever x_1 is close to x_2, $f(x_1)$ is close to $f(x_2)$.
- Function $f(x)$ is **increasing** on some interval if whenever $x_1 < x_2$, $f(x_1) < f(x_2)$ (so $f'(x)$ is positive). It is **decreasing** if $f(x_1) > f(x_2)$ (so $f'(x)$ is negative). A function that either never increases or never decreases is called **monotonic**.
- Function $f(x)$ is **differentiable** on some open interval if its derivative exists everywhere on that interval. A differentiable function must be continuous, and it cannot have vertical tangents on the interval.
- Function $f(x)$ is **concave up** on some interval if its second derivative $f''(x)$ is positive there; its graph "cups up." It is **concave down** if $f''(x)$ is negative; its graph "cups down."
- The line $x = a$ is a **vertical asymptote** for $f(x)$ if $f(x)$ "blows up" to (positive or negative) infinity as x gets closer and closer to a (from the left side, the right side, or both). Formally, $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ (or both).
- The line $y = b$ is a **horizontal asymptote** for $f(x)$ if the value of $f(x)$ gets close to b as $|x|$ becomes very large (when x is positive, negative, or both). Formally, $\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ (or both).

AREA UNDER A CURVE AND THE DEFINITE INTEGRAL

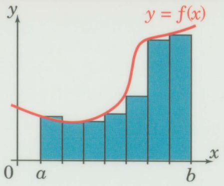
BASIC PROBLEM OF INTEGRAL CALCULUS

Given a function $y = f(x)$ on the interval $[a, b]$, what is the area enclosed by this curve, the x-axis, and the two vertical lines $x = a$ and $x = b$?

NOTE: We always speak of "signed" area: a curve above the x-axis is said to enclose positive area, while a curve below the x-axis is said to enclose negative area. The concept of "negative area" may seem ridiculous, but signed area is more versatile and simpler to keep track of.

APPROXIMATIONS TO THE AREA

- **Left-hand rectangle approximation:** We can approximate this area by a series of n rectangles. Divide the interval into n equal subintervals of width $\Delta x = \frac{b-a}{n}$ and obtain $n+1$ points on the x-axis at $x_0 = a$, $x_1 = a + \Delta x$, …, $x_n = a + n\Delta x = b$. These are the bottom corners of n rectangles, which we'll always number 0 to $n-1$. The height of each rectangle is the value of $f(x)$ at the left x-axis corner. The k^{th} rectangle has height $f(x_k)$ and area $\Delta x\, f(x_k)$. The total area of the n rectangles, then, is

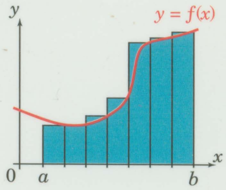


$$L_n = \Delta x\big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\big) = \Delta x \sum_{k=0}^{n-1} f(x_k)\,.$$

- Larger n will give more accurate approximation to the area.

Ex: We approximate the area under the curve $y = x^2$ on the interval $[0, 1]$ with 4 rectangles of width $\Delta x = \frac{1}{4}$ and heights $0, \left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2$ for a total area of
$L_4 = \frac{1}{4}\left((0)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{7}{32} = 0.21875.$

- **Right-hand rectangle approximation:** Instead of taking the height of each rectangle to be the value of $f(x)$ at the left x-axis corner, we can take the the value of $f(x)$ at the right corner. The height of the k^{th} rectangle is now $f(x_{k+1})$ for a total area of



$$R_n = \Delta x\big(f(x_1) + f(x_2) + \cdots + f(x_n)\big) = \Delta x \sum_{k=1}^{n} f(x_k)\,.$$

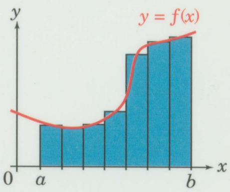
- Right-hand and left-hand approximations are related by $R_n = L_n + \Delta x\big(f(b) - f(a)\big)$.

Ex: For $f(x) = x^2$ on the interval $[0, 1]$,
$R_4 = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + (1)^2\right) = \frac{15}{32} = 0.46875.$

- **Midpoint Rule:** The height of each rectangle can be taken to be $f(x)$ evaluated at the midpoint of each rectangle; the height of the k^{th} rectangle is now

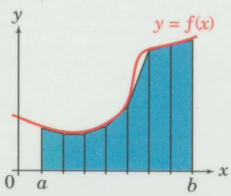
$$f\left(a + \Delta x\left(k + \tfrac{1}{2}\right)\right) = f\left(\tfrac{x_k + x_{k+1}}{2}\right)$$

for a total area of



$$M_n = \Delta x\left(f\left(\tfrac{x_0+x_1}{2}\right) + \cdots f\left(\tfrac{x_{n-1}+x_n}{2}\right)\right) = \Delta x \sum_{k=0}^{n-1} f\left(\tfrac{x_k + x_{k+1}}{2}\right).$$

- **Trapezoidal Rule:** We can approximate the area under the curve using trapezoids with the same two vertical sides and x-axis side as the rectangles. The area of a trapezoid is (*average length of two parallel sides*) × (*distance between them*). The area of the k^{th} trapezoid, then, is $\frac{\Delta x}{2}\left(f(x_k) + f(x_{k+1})\right)$, for a total area of



$$T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right).$$

- **Simpson's Rule:** This time, we suppose n to be even and approximate the area with $\frac{n}{2}$ parabola pieces with the k^{th} parabola defined by points on the curve at x_{2k-2}, x_{2k-1}, and x_{2k}. The total area is given by

$$S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right).$$

COMPARING APPROXIMATIONS TO THE AREA

- If $f(x)$ is increasing on the interval $[a, b]$, then $L_n <$ Area $< R_n$ for each n. If $f(x)$ is decreasing on the whole interval, the inequalities are reversed.
- If $f(x)$ is concave up on the whole interval, then L_n is a better approximation to the area than R_n. If $f(x)$ is concave down on the whole interval, then R_n is a better approximation than L_n.
- The Midpoint Rule approximates area more accurately than the Trapezoidal Rule. Both are better than the left- or right-hand rectangle approximations. Simpson's Rule is best of all.

RIEMANN SUMS

The left- and right-hand rectangle approximations and the Midpoint Rule all use a prescribed point in the subinterval as the height of the rectangle. In general, we can pick *any* **sample point** x_k^* in the k^{th} subinterval. The area approximation, then, is $\Delta x \sum_{k=0}^{n-1} f(x_k^*)$. This general area approximation is called a **Riemann sum,** and its limit as n increases will give the area of the region.

THE DEFINITE INTEGRAL

If the limit $\lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f(x_k^*)$ exists, then the function $f(x)$ is called **integrable** on the interval $[a, b]$. The limit represents the area under the curve and is denoted $\int_a^b f(x)\,dx$.

- In this notation, $\int$ is the **integral sign,** $f(x)$ is the **integrand,** and a and b are the **lower** and **upper limits of integration,** respectively.
- The marker dx keeps track of the variable of integration and evokes a very small Δx; intuitively, the "integral from a to b of $f(x)\,dx$" is a sum of heights (function values) times tiny widths dx (i.e., a sum of many minute areas).
- Being "integrable" says nothing about how easy the symbolic integral is to write down. Often, the integral is difficult to express.
- All functions made up of a finite number of pieces of continuous functions are integrable. In practice, every function encountered in a Calculus class will be integrable except at points where it "blows up" towards $\pm\infty$ (equivalently, has a vertical asymptote).

Properties of the definite integral: Let $f(x)$ and $g(x)$ be functions integrable on the interval $[a, b]$, and p be a point inside the interval.

1. Sums and differences: $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

2. Scalar multiples: $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx.$ Here, c is any real number.

3. Reversing the limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

4. Concatenation: $\int_a^p f(x)\,dx + \int_p^b f(x)\,dx = \int_a^b f(x)\,dx.$

5. Betweenness: If $f(x) \le g(x)$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

In particular:

- If $f(x) \ge 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$.
- If M is the maximum value of $f(x)$ on $[a, b]$ and m is the minimum value, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

ANTIDERIVATIVES AND THE INDEFINITE INTEGRAL

Antidifferentiation is the reverse of differentiation: an **antiderivative** of $f(x)$ is any function $F(x)$ whose derivative is equal to the original function: $F'(x) = f(x)$ in a pre-established region. Functions that differ by constants have the same derivative; therefore, we look for a **family of antiderivatives** $F(x) + C$, where C is any real constant.

The family of the antiderivatives of $f(x)$ is denoted by the **indefinite integral:**

$$\int f(x)\,dx = F(x) + C \text{ if and only if } F'(x) = f(x).$$

The indefinite integral represents a family of functions differing by constants.