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CALCULUS REFERENCE

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THEORY

DERIVATIVES AND DIFFERENTIATION

Definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

DERIVATIVE RULES

1. **Sum and Difference:** $\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$
2. **Scalar Multiple:** $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$
3. **Product:** $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$

   Mnemonic: If f is "hi" and g is "ho," then the product rule is "ho d hi plus hi d ho."
4. **Quotient:** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

   Mnemonic: "Ho d hi minus hi d ho over ho ho."
5. **The Chain Rule**
   - First formulation: $(f \circ g)'(x) = f'(g(x))\,g'(x)$
   - Second formulation: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
6. **Implicit differentiation:** Used for curves when it is difficult to express y as a function of x. Differentiate both sides of the equation with respect to x. Use the chain rule carefully whenever y appears. Then, rewrite $\frac{dy}{dx} = y'$ and solve for y'.

   Ex: $x\cos y - y^2 = 3x$. Differentiate to first obtain $\frac{dx}{dx}\cos y + x\frac{d(\cos y)}{dx} - 2y\frac{dy}{dx} = 3\frac{dx}{dx}$, and then $\cos y - x(\sin y)y' - 2yy' = 3$. Finally, solve for $y' = \frac{\cos y - 3}{x\sin y + 2y}$.

COMMON DERIVATIVES

1. **Constants:** $\frac{d}{dx}(c) = 0$
2. **Linear:** $\frac{d}{dx}(mx+b) = m$
3. **Powers:** $\frac{d}{dx}(x^n) = nx^{n-1}$ (true for all real $n \neq 0$)
4. **Polynomials:** $\frac{d}{dx}(a_nx^n + \cdots + a_2x^2 + a_1x + a_0) = a_nnx^{n-1} + \cdots + 2a_2x + a_1$
5. **Exponential**
   - **Base e:** $\frac{d}{dx}(e^x) = e^x$
   - **Arbitrary base:** $\frac{d}{dx}(a^x) = a^x \ln a$
6. **Logarithmic**
   - **Base e:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$
   - **Arbitrary base:** $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$
7. **Trigonometric**
   - **Sine:** $\frac{d}{dx}(\sin x) = \cos x$
   - **Cosine:** $\frac{d}{dx}(\cos x) = -\sin x$
   - **Tangent:** $\frac{d}{dx}(\tan x) = \sec^2 x$
   - **Cotangent:** $\frac{d}{dx}(\cot x) = -\csc^2 x$
   - **Secant:** $\frac{d}{dx}(\sec x) = \sec x \tan x$
   - **Cosecant:** $\frac{d}{dx}(\csc x) = -\csc x \cot x$
8. **Inverse Trigonometric**
   - **Arcsine:** $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
   - **Arccosine:** $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
   - **Arctangent:** $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
   - **Arccotangent:** $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
   - **Arcsecant:** $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
   - **Arccosecant:** $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

INTEGRALS AND INTEGRATION

DEFINITE INTEGRAL

The **definite integral** $\int_a^b f(x)\,dx$ is the **signed area** between the function $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

- **Formal definition:** Let n be an integer and $\Delta x = \frac{b-a}{n}$. For each $k = 0, 2, \ldots, n-1$, pick point x_k^* in the interval $[a + k\Delta x, a + (k+1)\Delta x]$. The expression $\Delta x \sum_{k=0}^{n-1} f(x_k^*)$ is a **Riemann sum.** The definite integral $\int_a^b f(x)\,dx$ is defined as $\lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f(x_k^*)$.

INDEFINITE INTEGRAL

- **Antiderivative:** The function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.
- **Indefinite integral:** The indefinite integral $\int f(x)\,dx$ represents a **family of antiderivatives:** $\int f(x)\,dx = F(x) + C$ if $F'(x) = f(x)$.

FUNDAMENTAL THEOREM OF CALCULUS

Part 1: If $f(x)$ is continuous on the interval $[a,b]$, then the area function $F(x) = \int_a^x f(t)\,dt$ is continuous and differentiable on the interval and $F'(x) = f(x)$.

Part 2: If $f(x)$ is continuous on the interval $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

APPROXIMATING DEFINITE INTEGRALS

1. **Left-hand rectangle approximation:** $L_n = \Delta x \sum_{k=0}^{n-1} f(x_k)$
2. **Right-hand rectangle approximation:** $R_n = \Delta x \sum_{k=1}^{n} f(x_k)$
3. **Midpoint Rule:** $M_n = \Delta x \sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2}\right)$
4. **Trapezoidal Rule:** $T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$
5. **Simpson's Rule:** $S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$

TECHNIQUES OF INTEGRATION

1. **Properties of Integrals**
   - **Sums and differences:** $\int \left(f(x) \pm g(x)\right)dx = \int f(x)\,dx \pm \int g(x)\,dx$
   - **Constant multiples:** $\int cf(x)\,dx = c\int f(x)\,dx$
   - **Definite integrals: reversing the limits:** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
   - **Definite integrals: concatenation:** $\int_a^p f(x)\,dx + \int_p^b f(x)\,dx = \int_a^b f(x)\,dx$
   - **Definite integrals: comparison:** If $f(x) \leq g(x)$ on the interval $[a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
2. **Substitution Rule**—a.k.a. ***u*-substitutions:** $\int f\left(g(x)\right)g'(x)\,dx = \int f(u)\,du$
   - $\int f\left(g(x)\right)g'(x)\,dx = F(g(x)) + C$ if $\int f(x)\,dx = F(x) + C$.
3. **Integration by Parts**

   Best used to integrate a product when one factor ($u = f(x)$) has a simple derivative and the other factor ($dv = g'(x)\,dx$) is easy to integrate.
   - **Indefinite Integrals:** $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$ or $\int u\,dv = uv - \int v\,du$
   - **Definite Integrals:** $\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$
4. **Trigonometric Substitutions:** Used to integrate expressions of the form $\sqrt{\pm a^2 \pm x^2}$.

Expression	Trig substitution	Expression becomes	Range of θ	Pythagorean identity used
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $dx = a\cos\theta\,d\theta$	$a\cos\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ $(-a \leq x \leq a)$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta\,d\theta$	$a\tan\theta$	$0 \leq \theta < \frac{\pi}{2}$ $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$
$\sqrt{x^2 + a^2}$	$x = a\tan\theta$ $dx = a\sec^2\theta\,d\theta$	$a\sec\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$

APPLICATIONS

GEOMETRY

Area: $\int_a^b \left(f(x) - g(x)\right)dx$ is the area bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ if $f(x) \geq g(x)$ on $[a,b]$.

Volume of revolved solid (disk method): $\pi\int_a^b \left(f(x)\right)^2 dx$ is the volume of the solid swept out by the curve $y = f(x)$ as it revolves around the x-axis on the interval $[a,b]$.

Volume of revolved solid (washer method): $\pi\int_a^b \left(f(x)\right)^2 - \left(g(x)\right)^2 dx$ is the volume of the solid swept out between $y = f(x)$ and $y = g(x)$ as they revolve around the x-axis on the interval $[a,b]$ if $f(x) \geq g(x)$.

Volume of revolved solid (shell method): $\int_a^b 2\pi x f(x)\,dx$ is the volume of the solid obtained by revolving the region under the curve $y = f(x)$ between $x = a$ and $x = b$ around the y-axis.

Arc length: $\int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$ is the length of the curve $y = f(x)$ from $x = a$ to $x = b$.

Surface area: $\int_a^b 2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\,dx$ is the area of the surface swept out by revolving the function $y = f(x)$ about the x-axis between $x = a$ and $x = b$.

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