Example

Suppose we seek $\displaystyle\int \frac{1}{4+9x^2}\,\mathrm{d}x$.

We proceed by first extracting a factor of 4 from the denominator:

$$\int \frac{1}{4+9x^2}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{1+\frac{9}{4}x^2}\,\mathrm{d}x$$

This is very close to the standard result in the previous keypoint except that the term $\frac{9}{4}$ is not really wanted. Let us observe the effect of making the substitution $u = \frac{3}{2}x$, so that $u^2 = \frac{9}{4}x^2$. Then $\mathrm{d}u = \frac{3}{2}\mathrm{d}x$ and the integral becomes

$$\begin{aligned}\frac{1}{4}\int \frac{1}{1+\frac{9}{4}x^2}\,\mathrm{d}x &= \,= \frac{1}{4}\int \frac{1}{1+u^2}\cdot\frac{2}{3}\,\mathrm{d}u \\ &= \frac{1}{6}\int \frac{1}{1+u^2}\,\mathrm{d}u\end{aligned}$$

This can be finished off using the standard result, to give $\displaystyle\frac{1}{6}\tan^{-1}u + c = \frac{1}{6}\tan^{-1}\frac{3}{2}x + c$.

We now consider a similar example for which a sine substitution is appropriate.

Example

Suppose we wish to find $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x$.

The substitution we will use here is based upon the observations that in the denominator we have a term $a^2 - x^2$, and that there is a trigonometric identity $1 - \sin^2 A = \cos^2 A$ (and hence $(a^2 - a^2\sin^2 A = a^2\cos^2 A)$.

We try $x = a\sin\theta$, so that $x^2 = a^2\sin^2\theta$. Then $\frac{dx}{d\theta} = a\cos\theta$ and $\mathrm{d}x = a\cos\theta\mathrm{d}\theta$. The integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x &= \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}a\cos\theta\mathrm{d}\theta \\ &= \int \frac{1}{\sqrt{a^2\cos^2\theta}}a\cos\theta\mathrm{d}\theta \\ &= \int \frac{1}{a\cos\theta}a\cos\theta\mathrm{d}\theta \\ &= \int 1\,\mathrm{d}\theta \\ &= \theta + c \\ &= \sin^{-1}\frac{x}{a} + c\end{aligned}$$

Hence $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \sin^{-1}\frac{x}{a} + c$.
This is another standard result.