Lecture Notes for Analog Electronics

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December, 1999

Class Notes 3

3 Circuit Analysis in Frequency Domain

We now need to turn to the analysis of passive circuits (involving EMFs, resistors, capacitors, and inductors) in frequency domain. Using the technique of the complex impedance, we will be able to analyze time-dependent circuits algebraically, rather than by solving differential equations. We will start by reviewing complex algebra and setting some notational conventions. It will probably not be particularly useful to use the text for this discussion, and it could lead to more confusion. Skimming the text and noting results might be useful.

3.1 Complex Algebra and Notation

Let $\tilde{V}$ be the complex representation of V. Then we can write

$$\tilde{V} = \Re(\tilde{V}) + \imath\Im(\tilde{V}) = Ve^{\imath\theta} = V\left[\cos\theta + \imath\sin\theta\right]$$

where $\imath = \sqrt{-1}$. V is the (real) amplitude:

$$V = \sqrt{\tilde{V}\tilde{V}^*} = \left[\Re^2(\tilde{V}) + \Im^2(\tilde{V})\right]^{1/2}$$

where $*$ denotes complex conjugation. The operation of determining the amplitude of a complex quantity is called taking the *modulus*. The phase θ is

$$\theta = \tan^{-1}\left[\Im(\tilde{V})/\Re(\tilde{V})\right]$$

So for a numerical example, let a voltage have a real part of 5 volts and an imaginary part of 3 volts. Then $\tilde{V} = 5 + 3\imath = \sqrt{34}e^{\imath\tan^{-1}(3/5)}$.

Note that we write the amplitude of $\tilde{V}$, formed by taking its modulus, simply as V. It is often written $|\tilde{V}|$. We will also use this notation if there might be confusion in some context. Since the amplitude will in general be frequency dependent, it will also be written as $V(\omega)$. We will most often be interested in results expressed as amplitudes, although we will also look at the phase.

3.2 Ohm's Law Generalized

Our technique is essentially that of the Fourier transform, although we will not need to actually invoke that formalism. Therefore, we will analyze our circuits using a single Fourier frequency component, $\omega = 2\pi f$. This is perfectly general, of course, as we can add (or integrate) over frequencies if need be to recover a result in time domain. Let our complex Fourier components of voltage and current be written as $\tilde{V} = Ve^{\imath(\omega t + \phi_1)}$ and $\tilde{I} = Ie^{\imath(\omega t + \phi_2)}$.

Now, we wish to generalize Ohm's Law by replacing $V = IR$ by $\tilde{V} = \tilde{I}\tilde{Z}$, where $\tilde{Z}$ is the (complex) impedance of a circuit element. Let's see if this can work. We already know that a resistor R takes this form. What about capacitors and inductors?

Our expression for the current through a capacitor, $I = C(dV/dt)$ becomes

$$\tilde{I} = C\frac{d}{dt}Ve^{\imath(\omega t + \phi_1)} = \imath\omega C\tilde{V}$$

This, of course, yields the same $|\tilde{V}_{\rm out}|$ as we found before in Eqn. 6 of Section 3.3. But now we also have included the phase information. The "real" time-dependent solution is then just the real part of this:

$$V_{\rm out}(t) = \Re(\tilde{V}_{\rm out}) = V_{\rm out}\cos(\omega t + \phi)$$

where ϕ is given by Eqn. 15.

3.7 Power in Reactive Circuits

Recall that for DC voltages and currents the power associated with a circuit element carrying current I with voltage change V is just $P = VI$. Now, for time-varying voltages and currents we have to be more careful. We could still define an instantaneous power as the product $V(t)I(t)$. However, it is generally more useful to average the power over time.

3.7.1 General Result for AC

Since we are considering Fourier components, we will average the results over one period $T = 1/f = 2\pi/\omega$. Therefore, the time-averaged power is

$$< P > = \frac{1}{T}\int_0^T V(t)I(t)dt$$

where the brackets indicate the time average. Let the voltage and current be out of phase by an arbitrary phase angle ϕ. So we have $V(t) = V_0\cos(\omega t)$ and $I(t) = I_0\cos(\omega t + \phi)$. We can plug these into the expression for $< P >$ and simplify using the following: $\cos(\omega t + \phi) = \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)$; $\int_0^T \sin(\omega t)\cos(\omega t)dt = 0$; and $(1/T)\int_0^T \sin^2(\omega t)dt = (1/T)\int_0^T \cos^2(\omega t)dt = 1/2$. This yields

$$< P > = \frac{1}{2}V_0 I_0\cos\phi = V_{\rm RMS}I_{\rm RMS}\cos\phi \qquad (16)$$

In the latter expression we have used the "root mean squared", or *RMS*, amplitudes. Using voltage as an example, the RMS and standard amplitudes are related by

$$V_{\rm RMS} \equiv \left[\frac{1}{T}\int_0^T V^2(t)dt\right]^{1/2} = \left[\frac{1}{T}\int_0^T V_0^2\cos^2(\omega t)dt\right]^{1/2} = V_0/\sqrt{2} \qquad (17)$$

3.7.2 Power Using Complex Quantities

Our results above can be simply expressed in terms of $\tilde{V}$ and $\tilde{I}$. Equivalent to above, we start with $\tilde{V}(t) = V_0 e^{\imath\omega t}$ and $\tilde{I}(t) = I_0 e^{\imath(\omega t+\phi)}$. By noting that

$$\Re(\tilde{V}^*\tilde{I}) = \Re\left(V_0 I_0(\cos\phi + \imath\sin\phi)\right) = V_0 I_0\cos\phi$$

we identify an expression for average power which is equivalent to Eqn. 16 :

$$< P > = \frac{1}{2}\Re(\tilde{V}^*\tilde{I}) = \frac{1}{2}\Re(\tilde{V}\tilde{I}^*) \qquad (18)$$

3.9 More Filters

3.9.1 Combining Filter Sections

Filter circuits can be combined to produce new filters with modified functionality. An example is the homework problem (6) of page 59 of the text, where a high-pass and a low-pass filter are combined to form a "band-pass" filter. As discussed at length in Section 1.5, it is important to design a "stiff" circuit, in which the next circuit element does not load the previous one, by requiring that the output impedance of the first be much smaller than the input impedance of the second. We can standardize this inequality by using a factor of 10 for the ratio $|\tilde{Z}_{\rm in}|/|\tilde{Z}_{\rm out}|$.

3.9.2 More Powerful Filters

This technique of cascading filter elements to produce a better filter is discussed in detail in Chapter 5 of the text. In general, the transfer functions of such filters take the form (for the low-pass case):

$$T(\omega) = \left[1 + \alpha_n (f/f_c)^{2n}\right]^{-1/2}$$

where f_c is the 3 db frequency, α_n is a coefficient depending upon the type of filter, and n is the filter "order," often equal to the number of filtering capacitors.

3.9.3 Active Filters

Filters involving LC circuits are very good, better than the simple RC filters, as discussed above. Unfortunately, inductors are, in practice, not ideal lumped circuit elements and are difficult to fabricate. In addition, filters made entirely from passive elements tend to have a lot of attenuation. For these reasons active filters are most commonly used where good filtering is required. These typically use operational amplifiers (which we will discuss later), which can be configured to behave like inductors, and can have provide arbitrary voltage gain. Again, this is discussed in some detail in Chapter 5. When we discuss op amps later, we will look at some examples of very simple active filters. At high frequencies (for example RF), op amps fail, and one most fall back on inductors.

4 Diode Circuits

The figure below is from Lab 2, which gives the circuit symbol for a diode and a drawing of a diode from the lab. Diodes are quite common and useful devices. One can think of a diode as a device which allows current to flow in only one direction. This is an over-simplification, but a good approximation.

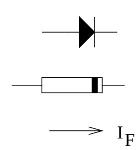

Figure 13: Symbol and drawing for diodes.

A diode is fabricated from a *pn* junction. Semi-conductors such as silicon or germanium can be "doped" with small concentrations of specific impurities to yield a material which conducts electricity via electron transport (n-type) or via holes (p-type). When these are brounght together to form a *pn* junction, electrons (holes) migrate away from the n-type (p-type) side, as shown in Fig. 14. This redistribution of charge gives rise to a potential gap ΔV across the junction, as depicted in the figure. This gap is $\Delta V \approx 0.7$ V for silicon and ≈ 0.3 V for germanium.

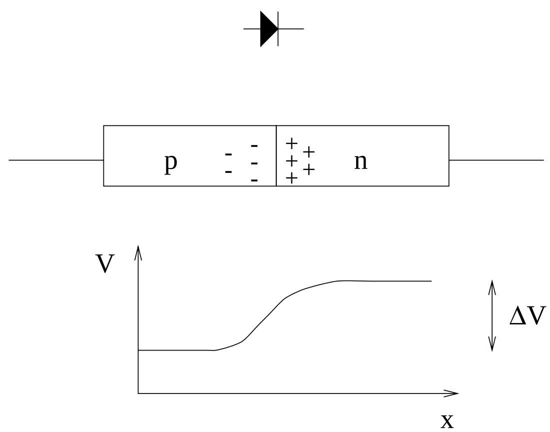

Figure 14: A *pn* junction, forming a voltage gap across the junction.

When a diode is now connected to an external voltage, this can effectively increase or decrease the potential gap. This gives rise to very different behavior, depending upon the polarity of this external voltage, as shown by the typical V-I plot of Fig. 15. When the diode is "reverse biased," as depicted in the figure, the gap increases, and very little current flows across the junction (until eventually at ~ 100 V field breakdown occurs). Conversely, a "forward biased" configuration decreases the gap, approaching zero for an external voltage equal to the gap, and current can flow easily. An analysis of the physics gives the form

$$I = I_S \left[e^{eV/kT} - 1 \right]$$

where I_S is a constant, V is the applied voltage, and $kT/e = 26$ mV at room temperature.

Thus, when reverse biased, the diode behaves much like an open switch; and when forward biased, for currents of about 10 mA or greater, the diode gives a nearly constant voltage drop of ≈ 0.6 V.

5.7 Transistor Differential Amplifier

Differential amplifiers are in general very useful. They consist of two inputs and one output, as indicated by the generic symbol in Fig. 24. The output is proportional to the *difference* between the two inputs, where the proportionality constant is the gain. One can think of this as one of the two inputs (labelled "−") being inverted and then added to the other non-inverting input (labelled "+"). Operational amplifiers ("op amps"), which we will soon study, are fancy differential amplifiers, and are represented by the same symbol as that of Fig. 24.

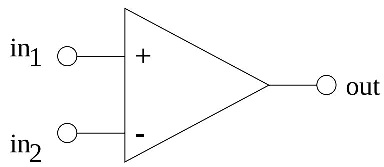

Figure 24: Symbol for a differential amplifier or op amp.

This technique is commonly used to mitigate noise pickup. For example, a signal which is to be transmitted and subject to noise pickup can first be duplicated and inverted. This "differential pair" is then transmitted and then received by a differential amplifier. Any noise pickup will be approximately equal for the two inputs, and hence will not appear in the output of the differential amplifier. This "common mode" noise is rejected. This is often quantified by the common-mode rejection ratio (CMRR) which is the ratio of differential gain to common-mode gain. Clearly, a large CMRR is good.

5.7.1 A Simple Design

The circuit shown in Fig. 25 represents a differential amplifier design. It looks like two common-emitter amplifiers whose emitters are tied together at point A. In fact, the circuit does behave in this way. It is simplest to analyze its output if one writes each input as the sum of two terms, a sum and a difference. Consider two signals v_1 and v_2. In general, we can rewrite these as $v_1 = <v> + \Delta v/2$ and $v_2 = <v> - \Delta v/2$, where $<v> = (v_1 + v_2)/2$ is the average and $\Delta v = v_1 - v_2$ is the difference. Therefore, we can break down the response of the circuit to be due to the response to a common-mode input ($<v>$) and a difference (Δv) input.

Let's analyze the difference signal first. Therefore, consider two inputs $v_1 = \Delta v/2$ and $v_2 = -\Delta v/2$. The signals at the emitters then follow the inputs, as usual, so that at point A we have $v_A = v_{E1} + v_{E2} = v_1 + v_2 = 0$. Following the common-emitter amplifier derivation, we have $v_{\text{out1}} = -i_C R_C$, where $i_C \approx i_E = v_E/R_E = v_{\text{in1}}/R_E$. Hence, $v_{\text{out1}} = -(R_C/R_E)v_1$ and $v_{\text{out2}} = -(R_C/R_E)v_2$. We define the *differential gain* G_{diff} as the ratio of the output to the input difference. So

$$G_{\text{diff1}} \equiv v_{\text{out1}}/\Delta v = -(R_C/R_E)v_1/(2v_1) = -R_C/(2R_E)$$

Hence, variations in β are attenuated by the factor $\beta + 1$. So this represents a good design.

The variation in the output of this current source resulting from the Early effect can be evaluated similarly:

$$\frac{\Delta I_L}{I_L} = \frac{1}{I_L}\frac{dI_L}{dV_{\mathrm{BE}}}\Delta V_{\mathrm{BE}} = -\frac{\Delta V_{\mathrm{BE}}}{V_B - V_{\mathrm{BE}}} = \frac{1\times 10^{-4}}{V_B - V_{\mathrm{BE}}}\Delta V_{\mathrm{CE}}$$

which can be evaluated using the compliance range for ΔV_{CE}.

Temperature dependence can now be estimated, as well. Using our current source, again, to exemplify this point, we see that temperature dependence can show up both in V_{BE} and β. The former effect can be evaluated using the chain rule and the result from the previous paragraph:

$$\frac{dI_L}{dT} = \frac{dI_L}{dV_{\mathrm{BE}}}\frac{dV_{\mathrm{BE}}}{dT} \approx \frac{2.1\ \mathrm{mV}/^\circ\mathrm{C}}{R_E}$$

Therefore, we see that temperature dependence is $\propto 1/R_E$. As before, R_E is in general replaced by the sum $R_E + r_e$. In the case where the external resistor is omitted, then the typically small r_e values can induce a large temperature dependence (*cf* problem 7 at the end of Chapter 2 of the text). Similarly, using previous results, we can estimate the effect of allowing $\beta = \beta(T)$:

$$\frac{dI_L}{dT} = \frac{dI_L}{d\beta}\frac{d\beta}{dT} = \frac{I_L}{\beta+1}\left(\frac{1}{\beta}\frac{d\beta}{dT}\right)$$

where the term in parentheses, the fractional gain temperature dependence, is often a known parameter (*cf* problem 2d at the end of Chapter 2 of the text).

2. The inputs draw no current.
   (This is true in the approximation that the Z_{in} of the op-amp is much larger than any other current path available to the inputs.)

When we assume ideal op-amp behavior, it means that we consider the golden rules to be exact. We now use these rules to analyze the two most common op-amp configurations.

6.2 Inverting Amplifier

The inverting amplifier configuration is shown in Fig. 30. It is "inverting" because our signal input comes to the "−" input, and therefore has the opposite sign to the output. The negative feedback is provided by the resistor R_2 connecting output to input.

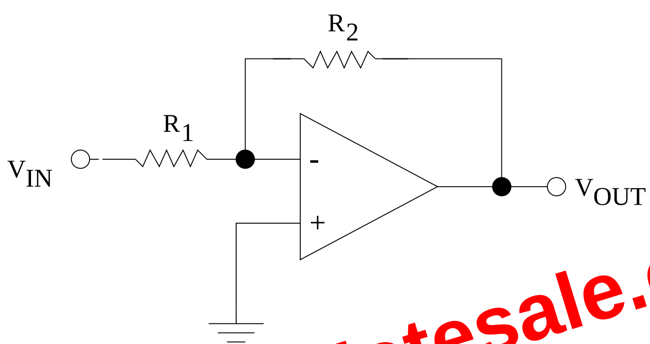

Figure 30: Inverting amplifier configuration.

We can use our rules to analyze this circuit. Since input + is connected to ground, then by rule 1, input − is also at ground. For this reason, the input − is said to be at *virtual ground*. Therefore, the voltage drop across R_1 is $v_{\text{in}} - v_- = v_{\text{in}}$, and the voltage drop across R_2 is $v_{\text{out}} - v_- = v_{\text{out}}$. So, applying Kirchoff's first law to the node at input −, we have, using golden rule 2:

$$i_- = 0 = i_{\text{in}} + i_{\text{out}} = v_{\text{in}}/R_1 + v_{\text{out}}/R_2$$

or

$$G = v_{\text{out}}/v_{\text{in}} = -R_2/R_1 \tag{34}$$

The input impedance, as always, is the impedance to ground for an input signal. Since the − input is at (virtual) ground, then the input impedance is simply R_1:

$$Z_{\text{in}} = R_1 \tag{35}$$

The output impedance is very small ($< 1\ \Omega$), and we will discuss this again soon.

6.3 Non-inverting Amplifier

This configuration is given in Fig. 31. Again, its basic properties are easy to analyze in terms of the golden rules.

$$v_{\text{in}} = v_+ = v_- = v_{\text{out}} \left[\frac{R_1}{R_1 + R_2} \right]$$

Therefore, the efect of the closed loop circuit is to improve both input and output impedances by the identical loop-gain factor $1 + AB \approx AB$. So for a typical op-amp like a 741 with $A = 10^3$, $R_i = 1$ MΩ, and $R_o = 100$ Ω, then if we have a loop with $B = 0.1$ we get $Z_{\text{in}} = 100$ MΩ and $Z_{\text{out}} = 1$ Ω.

6.6.3 Examples of Negative Feedback Benefits

We just demonstrated that the input and output impedance of a device employing negative feedback are both improved by a factor $1 + AB \approx AB$, the device loop gain. Now we give a simple example of the gain equation Eqn. 42 in action.

An op-amp may typically have an open-loop gain A which varies by at least an order of magnitude over a useful range of frequency. Let $A_{\max} = 10^4$ and $A_{\min} = 10^3$, and let $B = 0.1$. We then calculate for the corresponding closed-loop gain extremes:

$$G_{\max} = \frac{10^4}{1 + 10^3} \approx 10(1 - 10^{-3})$$

$$G_{\min} = \frac{10^3}{1 + 10^2} \approx 10(1 - 10^{-2})$$

Hence, the factor of 10 open-loop gain variation has been reduced to a 1% variation. This is typical of negative feedback. It attenuates errors which appear within the feedback loop, either internal or external to the op-amp proper.

In general, the benefits of negative feedback go as the loop gain factor AB. For most op-amps, A is very large, starting at $> 10^5$ for $f < 100$ Hz. A large gain G can be achieved with large A and relatively small B, at the expense of somewhat poorer performance relative to a smaller gain, large B choice, which will tend to very good stability and error compensation properties. An extreme example of the latter choice is the "op-amp follower" circuit, consisting of a non-inverting amplifier (see Fig. 31) with $R_2 = 0$ and R_1 removed. In this case, $B = 1$, giving $G = A/(1 + A) \approx 1$.

Another interesting feature of negative feedback is one we discussed briefly in class. The qualitative statement is that any signal irregularity which is put into the feedback loop will, in the limit $B \to 1$, be taken out of the output. This reasoning is as follows. Imagine a small, steady signal v_s which is added within the feedback loop. This is returned to the output with the opposite sign after passing through the feedback loop. In the limit $B = 1$ the output and feedback are identical ($G = 1$) and the cancellation of v_s is complete. An example of this is that of placing a "push-pull" output stage to the op-amp output in order to boost output current. (See text Section 2.15.) The push-pull circuits, while boosting current, also exhibit "cross-over distortion", as we discussed in class and in the text. However, when the stage is placed within the op-amp negative feedback loop, this distortion can essentially be removed, at least when the loop gain AB is large.

6.7 Compensation in Op-amps

Recall that an RC filter introduces a phase shift between 0 and $\pi/2$. If one cascades these filters, the phase shifts can accumulate, producing at some frequency ω_π the possibility of a phase shift of $\pm\pi$. This is dangerous for op-amp circuits employing negative feedback, as a phase shift of π converts negative feedback to *positive* feedback. This in turn tends to